Tentamen Toegepaste Logica

16 januari 2002

Answers may be given in Dutch or English. Good luck!

Exercise 1.

a. Show that the formula $(A \to A \to B) \to A \to C \to B$ is a tautology of first-order minimal propositional logic. (Give a proof in natural deduction with all assumptions canceled.)

   (5 points)

b. Give the (formal) type derivation in simply typed λ-calculus corresponding to the proof of 1a.

   (5 points)

c. Give a natural deduction proof of $A \to (B \to A) \to A$ with a detour and with all assumptions canceled.

   (5 points)

d. Give the typing derivation corresponding to the proof of 1c, and reduce the term to β-normal form.

   (5 points)

Exercise 2.

a. What is the interpretation of a sequent $A_1, \dots, A_m \vdash B_1, \dots, B_n$?

   (2 points)

b. Explain the intuition of the left rule for existential quantification in sequent calculus:

$$\frac{\Gamma, A[x := y] \vdash \Delta}{\Gamma, \exists x.\, A \vdash \Delta}\ L\exists$$

   with y a fresh variable (not occurring in Γ, Δ).

   (3 points)

c. Give a derivation of $\vdash A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$.

(5 points)

d. For first-order propositional sequent calculus in PVS, we use two commands: `flatten`, corresponding to the rules of the sequent calculus where reading upwards a sequent is transformed into one sequent, and `split`, corresponding to the rules of the sequent calculus where reading upwards a sequent is transformed into two sequents.

Give the proof tree for the PVS derivation of
$\vdash A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$.

(You may give a refinement of the PVS proof tree, using more steps where PVS uses only one step.)

(5 points)

Exercise 3.

a. In PVS specifications can be written using the axiomatic and the definitional approach. Explain the two approaches and explain a drawback of each of them.

(5 points)

b. Consider the abstract datatype specification for stacks:

```
stack [t:TYPE] : DATATYPE
BEGIN

empty : emptystack?
push(top:t, pop:stack) : nonemptystack?

END stack
```

Explain the notions of constructors, recognizers, and accessors.

(5 points)

c. Explain the use of predicate subtypes in PVS.

(5 points)

Exercise 4.

a. Consider the rule for introduction of the universal quantifier:

$$\frac{A}{\forall x.\, A}\ I\forall$$

with x not in uncanceled assumptions.

Explain the side condition and give an example that shows what can go wrong.

(5 points)

b. Show that the formula $(\forall x. \forall y. (R(x, y) \to \neg R(y, x))) \to \forall z. \neg R(z, z)$ is a tautology of first-order predicate logic. (Give a derivation in (intuitionistic) natural deduction with all assumptions canceled.)

(5 points)

Exercise 5.

a. Give the definition of the inductive type `nat` that is built in in Coq.

(5 points)

b. Give your own definition in Coq of an inductive type `natpair : Set`, representing the set of pairs of natural numbers. (Hint: use one constructor `Pair`; the pair of n and m is then denoted in Coq by `(Pair n m)`.)

(5 points)

c. What is the type of a predicate (i.e. a propositional function) on `natpair`?

(5 points)

d. Give an inductive definition in Coq of the predicate `ordered` on `natpair`, such that we have in Coq that `(ordered (Pair n m))` if and only if `(leq n m)`, with `leq` the 'less than or equal' predicate that is built in in Coq.

(5 points)

Exercise 6. This exercise is concerned with $\lambda 2$, the polymorphic λ-calculus.

Consider the operation *NatTwice* that, when applied to a function f of type Nat $\to$ Nat forms the composition of f with itself. So we have

$$\mathit{NatTwice}\ f\ n = f(f\ n)$$

a. Give an explicit definition of *NatTwice* as a term of the simply typed λ-calulus, and give its type.

(5 points)

b. Give a definition of *PolyTwice*, the polymorphic version of *NatTwice*, as a term of $\lambda 2$, and give its type.

(5 points)

The final note is (the total amount of points plus 10) divided by 10.

Appendix: rules of first-order propositional sequent calculus

$$\Gamma, A \vdash A, \Delta$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ R\to$$

$$\frac{B, \Gamma \vdash \Delta \qquad \Gamma \vdash A, \Delta}{A \to B, \Gamma \vdash \Delta}\ L\to$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ R\wedge$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ L\wedge$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ R\vee$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ L\vee$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ R\neg$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ L\neg$$

$$\frac{\Gamma_1 \vdash \Delta_1}{\Gamma_2 \vdash \Delta_2}\ w$$

(with $\Gamma_1 \subseteq \Gamma_2$ and $\Delta_1 \subseteq \Delta_2$)

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}\ c$$