Tentamen Toegepaste Logica

17 januari 2001

Answers may be given in Dutch or English. Good luck!

Exercise 1. This exercise is concerned with first-order minimal propositional logic and $\lambda^{\rightarrow}$ (simply typed λ-calculus).

a. Show that the formula $((A \rightarrow B \rightarrow A) \rightarrow B) \rightarrow B$ is a tautology of first-order minimal propositional logic.

   (5 points)

b. Give the (formal) type derivation in simply typed λ-calculus corresponding to the proof of 1a.

   (5 points)

c. Give a proof of the formula $A \rightarrow B \rightarrow A$ with a detour, and with all assumptions cancelled.

   (5 points)

d. Give the typing derivation corresponding to the proof of 1c, and reduce the term to β-normal form.

   (5 points)

Exercise 2. This exercise is concerned with first-order minimal predicate logic and λP (λ-calculus with dependent types).

a. Show that the formula $(\forall x.(P(x) \rightarrow \forall y.Q(y))) \rightarrow \forall z.\forall w.(P(z) \rightarrow Q(w))$ is a tautology of first-order minimal predicate logic.

   (5 points)

b. Give the typing derivation in λP that corresponds exactly to the proof of 2a.

   (5 points)

c. Assume that an algebraic term is encoded by a term in Terms in λP. Explain the encoding of formulas of first-order minimal predicate logic in λP in detail.

   (5 points)

d. The formal typing system for λP contains the conversion rule:

$$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : \star/\Box}{\Gamma \vdash A : B'} \qquad \text{with } B =_\beta B'$$

Explain why this rule is necessary.

(5 points)

Exercise 3. This exercise is concerned with second-order minimal propositional logic and $\lambda 2$ (λ-calculus with polymorphic types).

a. Give the identity function on Bool and give the polymorphic identity function in $\lambda 2$.

What is the formula in second-order minimal propositional logic corresponding to the type of the polymorphic identity?

(5 points)

b. Explain the encoding of formulas of second-order minimal propositional logic in $\lambda 2$ in detail.

(5 points)

c. Consider the two product rules of $\lambda 2$ from its formal typing system:

$$\frac{\Gamma \vdash A : \star \qquad \Gamma, x : A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.\, B : \star} \qquad \frac{\Gamma \vdash A : \Box \qquad \Gamma, x : A \vdash B : \star}{\Gamma \vdash \Pi x{:}A.\, B : \star}$$

Do for each rule the following: explain briefly what it means and illustrate its use by means of an example.

(5 points)

Exercise 4. This exercise is concerned with Gödel's **T**. Besides the β-reduction rule we also use the rules for the conditional:

$$\begin{array}{lcl} \mathsf{c}(M, N, \mathsf{t}) & \to & M \\ \mathsf{c}(M, N, \mathsf{f}) & \to & N \end{array}$$

Here we take $M : \mathsf{Bool}$ and $N : \mathsf{Bool}$ and hence also $\mathsf{c}(M, N, \mathsf{t}) : \mathsf{Bool}$ and $\mathsf{c}(M, N, \mathsf{f}) : \mathsf{Bool}$.

a. Let $\mathsf{or} = \lambda x{:}\mathsf{Bool}.\, \lambda y{:}\mathsf{Bool}.\, \mathsf{c}(\mathsf{t}, y, x)$. Show that $(\mathsf{or}\, \mathsf{t}\, z) \to_T^* \mathsf{t}$ (give the reduction step by step).

(5 points)

b. Explain why the term $(\mathsf{or}\, z\, \mathsf{t})$ does *not* reduce to t.

(5 points)

c. Define a term or' such that $(\mathsf{or}'\, z\, \mathsf{t}) \to_T^* \mathsf{t}$ (give the reduction step by step).

(5 points)

Exercise 5.

a. Explain briefly what proof checking is and how it works.

   (5 points)

b. Coq is based on intuitionistic (also called constructive) logic. Give three different ways to extend *intuitionistic* first-order minimal propositional logic to *classical* first-order minimal propositional logic.

   (5 points)

Exercise 6. This question is concerned with inductive types.

a. Give and explain the definition of an inductive type `nat` of natural numbers in Coq.

   (5 points)

b. Give and explain the type of the term `nat_ind` that is automatically generated by Coq once the inductive definition of `nat` is given.

   (5 points)

Het tentamencijfer is (het totaal aantal punten plus 10) gedeeld door 10.