Exercise 1. (4 + 4)
Prove or disprove CR for the following two TRSs.

(a) $f(x, g(x)) \to f(x, x)$
$f(x, x) \to g(x)$

(b) $g(g(g(x))) \to g(x)$

Exercise 2. (3 + 3 + 3 + 3)
Recall that an ARS has the *normal form property* NF if for any normal form n and every a we have $a = n \Rightarrow a \twoheadrightarrow n$. (NB: $=$ stands for conversion.). Moreover an ARS has the *unique normal form property* UN if for normal forms a, b we have $a = b \;\Rightarrow\; a \equiv b$.
Prove or disprove the following implications:

(a) NF $\Rightarrow$ CR

(b) NF $\wedge$ WN $\Rightarrow$ CR

(c) UN $\wedge$ WN $\Rightarrow$ CR

(d) UN $\wedge$ SN $\Rightarrow$ CR

Exercise 3. (10)
Prove SN for the TRS with two rewrite rules:

$$\begin{array}{rcl} g(f(x), y) & \to & g(x, g(x, y)) \\ f(g(x, y)) & \to & g(f(x), g(f(y), x)) \end{array}$$

Exercise 4. (10 + 4 + 2)

(a) Use the Knuth–Bendix method to give a complete TRS for the following specification.

$$\begin{array}{rcl} A(A(x)) & = & A(x) \\ B(B(A(x))) & = & B(x) \end{array}$$

TIP: The system can be seen as a string rewriting system by dropping the brackets and the x. This may save writing.

(b) Prove that the resulting TRS is actually complete.

(c) Does the equation $A(B(x)) = B(A(x))$ hold in this specification? Motivate the answer.

Exercise 5. $(4 + 4 + 4 + 4)$
Consider the TRS with the two rewrite rules $a \to i(a)$ and $i(x) \to x$

(a) What is the term i^ω? Give an infinite reduction from a to i^ω of length ω.

(b) Give also a transfinite reduction from a to i^ω of length $\omega + 3$.

(c) Is there an infinite normal form n of the term a?

(d) Can you give a reason why the TRS will be CR^∞?

Exercise 6. $(4 + 4 + 4 + 4)$
This exercise is about CL.

(a) Give a reduction of the term $(IS)(IS)(IS)(IS)$ to normal form.

(b) Give a complete development of the term $(IS)(IS)(IS)(IS)$.

(c) Give a CL-term W such that $Wxy \twoheadrightarrow x(yy)$.

(d) Assume that N is a CL-term such that $N(xy) \twoheadrightarrow y$.

  (i) Show that then $N(K(II)K) \twoheadrightarrow K$ and also $N(K(II)K) \twoheadrightarrow I$. (Which general property of reduction do you use here?)

  (ii) Using (i) argue that such an N can not exist in CL.

Exercise 7. $(4 + 4 + 4)$
Consider the TRS

$$\begin{array}{rcl} f(x,y) & \to & g(y) \\ a & \to & h(a) \\ g(x) & \to & x \\ b & \to & c \end{array}$$

(a) Evaluate the term $g(f(a,b))$ to normal form with the full-substitution strategy $\mathbb{F}_{GK}$.

(b) Idem, now with the leftmost-outermost strategy $\mathbb{F}_{lm}$.

(c) Show what the parallel-innermost strategy does with the term $g(f(a,b))$.

The result is computed as the total number of points plus 10, divided by 10.