Exam Stochastic Processes

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1. Let W_t be standard Brownian motion and let $T > 0$ be a constant.
(a) Prove the reflection principle, that is, prove that $\tilde{W}_t$ defined by

$$\tilde{W}_t = \begin{cases} W_t & \text{for } t \leq T \\ 2W_T - W_t & \text{for } t > T, \end{cases}$$

is also a standard Brownian motion.
(b) Explain in words why we can replace the time T in (a) by a stopping time.
(c) Show that $\tau_a = \inf\{t \geq 0; W_t = a\}$ is a stopping time.
(d) Show that

$$P(W_t \geq x) = 1 - \Phi\left(\frac{x}{\sqrt{t}}\right),$$

where Φ denotes the distribution function of the standard normal distribution.

Now let $a > x > 0$ and $M_t = \max_{0 \leq s \leq t} W_t$.
(e) Show that, using (b) with stopping time τ_a, that

$$P(M_t \geq a, W_t \leq x) = 1 - \Phi\left(\frac{2a - x}{\sqrt{t}}\right).$$

2. Consider the unit interval $I = [0, 1]$, equipped with the usual sigma-algebra and Lebesgue measure. Let f be an integrable function on I. Let, for $n = 1, 2, \ldots$

$$f_n(x) = 2^n \int_{(k-1)2^{-n}}^{k2^{-n}} f(y)dy, \text{ for } (k-1)2^{-n} \leq x < k2^{-n},$$

and define $f_n(1) = 1$. (The value of $f_n(1)$ is not important.) Finally, we define $\mathcal{F}_n$ as the sigma algebra generated by intervals of the form $[(k-1)2^{-n}, k2^{-n})$, $1 \leq k < 2^n$.
(a) Argue that $\mathcal{F}_n$ is an increasing sequence of sigma-algebra's.
(b) Show that (f_n) is a martingale.
(c) Use Lévy's Upward Theorem to prove that as $n \to \infty$, $f_n \to f$, almost surely and in L_1.

3. Let $X_1, X_2, \ldots, X_n$ be independent uniform $[0,1]$ distributed random variables. We denote by $\mathbf{1}_A$ the indicater function of the event A, that is, $\mathbf{1}_A = 1$ if A occurs and 0 otherwise. For $0 \le t < 1$, define

$$G_n(t) = n^{-1} \sum_{k=1}^{n} \mathbf{1}_{\{X_k \le t\}},$$

in words; $G_n(t)$ is the fraction of the X_k's that has value at most t. We denote by $\mathcal{G}_n(t)$ the sigma-algebra $\sigma(G_n(s); s \le t)$.
(a) Explain why for $0 \le t < u \le 1$ we have

$$E(G_n(u)|\mathcal{G}_n(t)) = G_n(t) + [1 - G_n(t)]\frac{u-t}{1-t}.$$

(b) Use (a) to show that

$$M_n(t) = \frac{G_n(t) - t}{1-t}$$

is a continuous-time martingale with respect to $\{\mathcal{G}_n(t)\}$.
(c) Is $M_n(t)$ a uniform integrable martingale? (Hint: observe that $M_n(t)$ will be 1 for t close to 1.)

4. Let X_t be a continuous time Markov process on $\mathbb{Z}$ with the following transition rates: $q_{0,1} = \gamma$; for $i \ge 1$ we have $q_{i,i+1} = \lambda$ and $q_{i,i-1} = \mu$, with $\lambda + \mu = 1$ and $\mu > \lambda$.
(a) Write down the jump matrix of X_t. Why is this jump matrix independent of γ?

Denote the jump chain by $X_0, X_1, \ldots$, that is, X_n is the position after n jumps. Define

$$Y_n = \left(\frac{\mu}{\lambda}\right)^{X_n}$$

and $\tau_i = \min_n\{X_n = i\}$.
(b) We start the process at a point $m > 0$. Show that $Y_n^{\tau_0}$, that is, the Y_n process stopped at 0, is a martingale.
(c) Now start the process at a point m satisfying $0 < m < N$. Use the optional stopping theorem (verify the conditions!) to calculate the probability that the process hits N before it hits 0.

5. Suppose that $\Omega = \{+1, -1\}$, and that P is a probability measure with $P(\{+1\}) = P(\{-1\}) = 1/2$. Let $\mathcal{G}_t = \{\emptyset, \Omega\}$ when $t \le 1$ and $\mathcal{G}_t = \{\emptyset, \Omega, \{+1\}, \{-1\}\}$ when $t > 1$. Finally we define, for $\omega \in \Omega$,

$$Y_t(\omega) = \begin{cases} 0 & \text{if } t \le 1, \\ \omega & \text{if } t > 1. \end{cases}$$

(a) Show that (Y_t) is a martingale with respect to $\{\mathcal{G}_t\}$.

We define $X_t(\omega) = \lim_{s \downarrow t} Y_s(\omega)$.
(b) Show that (X_t) is not a martingale with respect to $\{\mathcal{G}_t\}$.
(c) Show that (X_t) is not a modification of (Y_t).