Exam Stochastic Processes

july 4 2005, 14.00-17.00

1. Let $(W_t)_{t\geq 0}$ be a standard Brownian motion with respect to its own filtration $\mathcal{F}_t$, and define, for $a > 0$,

$$S_a = \inf\{t \geq 0; W_t > a\}.$$

(a) Is S_a an optional time? Justify your answer.

Next we define T_a as the first time that W_t hits a, that is,

$$T_a = \inf\{t \geq 0; W_t = a\}.$$

Finally, let M_a be the *last* time that W_t is equal to at, that is,

$$M_a = \sup\{t \geq 0; W_t = at\}.$$

(b) Is T_a a stopping time? Is M_a a stopping time? Show that $M_a < \infty$ with probability one. (For this and the following question, you could use the fact that $tW_{1/t}$ is again a standard Brownian motion.)
(c) Show that M_a has the same distribution as $1/T_a$.
(d) Explain why the fact that $E(W_{T_a}|\mathcal{F}_0) \neq W_0$ does not contradict the optional stopping theorem.

2. Let $(M_t)_{0\leq t\leq\infty}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$, adapted to the filtration $\mathcal{F}_t$. *We do not assume that M_t is a martingale.* Assume that for any finite or infinite stopping time S, we have $E(|M_S|) < \infty$ and $E(M_S) = 0$. For $t \geq 0$ and $F \in \mathcal{F}_t$, we define $T(\omega) = t$ if $\omega \in F$, and $T(\omega) = \infty$ if $\omega \notin F$.
(a) Show that T is a stopping time.
(b) Show that

$$\int_F M_t dP + \int_{F^c} M_\infty dP = 0$$

and

$$\int_F M_\infty dP + \int_{F^c} M_\infty dP = 0,$$

where F^c denotes the complement of F.
(c) Deduce from (b) that

$$\int_F M_\infty dP = \int_F M_t dP.$$

(d) Deduce from (c) that $M_t = E(M_\infty | \mathcal{F}_t)$.
(e) Deduce from (d) and some results we have proved in class, that M_t is a uniformly integrable martingale. (Make sure that by now you have used all assumptions mentioned in this exercise.)

3. Consider a shop with one server. Customers arrive at the shop with rate $\lambda > 0$, that is, the time intervals between two arrivals are independent random variables with an exponential distribution with parameter λ. It takes an exponential amount of time (with parameter μ) to serve a customer, and these service times are independent of each other. The customers are served one by one in the order of arrival.
(a) Model this as a continuous time Markov process $(X_t)_{t \geq 0}$, where X_t represents the number of customers in the shop, and write down the Q-matrix.
(b) Show that

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i, \quad i = 0, 1, \ldots$$

is an invariant measure.
(c) Assume now that $\lambda < \mu$ and derive the invariant probability distribution for X_t.
(d) Explain why it is reasonable to have an invariant probability distribution for $\lambda < \mu$ but not for $\lambda > \mu$. What would happen when $\lambda > \mu$?

4. Consider modelling a sequence of gambling games by the following procedure. Let $M_n - M_{n-1}$ denotes the net winning per unit bet on game n, $n = 1, 2, \ldots$; this 'winning' can be negative of course. Let C_n be the amount you bet on game n. So the net winning on game n is $C_n(M_n - M_{n-1})$, and the total net winning up to game N is given by

$$(C \cdot M)_N = \sum_{n=1}^{N} C_n(M_n - M_{n-1}).$$

(a) Explain why C_n should be predictable and why M_n should be a martingale to define a reasonable model for a series of fair gambling games.

Now we become more specific. Consider a monkey that types a capital letter at random, at each of times $1, 2, \ldots$. Each letter of the 26 letters has the same probability, and different letters are independent of each other. Just before time $n = 1, 2, \ldots$, a new gambler arrives and bets 1 that the next letter (i.e. the n-th letter) is an A. If he loses he leaves the game, if

he wins, he gains 26 and bets all this on the event that the next letter is a B. This is repeated throughout the sequence ABRACADABRA. (Hence if the gambler wins the second time as well, his third bet is on the letter R, etcetera.) In questions (b)-(d) below we consider the first gambler.
(b) Show that the game the first gambler plays, can be formulated in the framework described at the beginning of this exercise, where $M_n - M_{n-1} = 25$ with probability $1/26$; $M_n - M_{n-1} = -1$ with probability $25/26$, and

$$C_n = 26^{n-1}$$

if the gambler is still in the game at time n, and $C_n = 0$ otherwise.
(c) Show that $(C \cdot M)_n = 26^n - 1$ if the gambler is succesful up to the n-th game, and $(C \cdot M)_n = -1$ otherwise.

Now let T be the first time that the monkey has typed ABRACADABRA. We are interested in the value of $E(T)$.
(d) Explain why $E(C \cdot M)_T = 0$.
(e) Show now, using (c) and (d), that $E(T) = 26^{11} + 26^4 + 26$ by combining all gamblers up to time T. (First figure out which gamblers are still in the game after the T-th gamble.)