Exam Stochastic Processes

june 13, 2005, 14.00-17.00

1. Let W_t $(t \geq 0)$ be Brownian motion, and define

$$X_t = W_t + ct,$$

for some constant c. The process X_t is called *Brownian motion with drift.* Fix some $\lambda > 0$.
(a) Show that

$$M_t := e^{\theta X_t - \lambda t}$$

is a martingale (with respect to the usual filtration) if and only if $\theta = \sqrt{c^2 + 2\lambda} - c$ or $\theta = -\sqrt{c^2 + 2\lambda} - c$. You can use the fact that

$$e^{aW_t - \frac{1}{2}a^2 t}$$

is a martingale for every real a.

Now let $x > 0$ and define $H_x = \inf\{t > 0; X_t = x\}$.
(b) Argue that H_x is a stopping time.
(c) Use the optional stopping theorem (verify all necessary conditions) to show that

$$E\left(e^{-\lambda H_x}\right) = e^{-x(\sqrt{c^2+2\lambda}-c)}.$$

(d) Use this to prove that

$$P(H_x < \infty) = 1 \text{ for } c \geq 0,$$

and

$$P(H_x < \infty) = e^{-2|c|x} \text{ for } c < 0.$$

(e) Explain why this result is reasonable.

2. Suppose that $Y_1, Y_2, \ldots$ are independent, positive random variables and that $E(Y_n) = 1$ for all n. Let $X_n = Y_1 \cdot Y_2 \cdots Y_n$.
(a) Show that (X_n) is a martingale and that X_n converges with probability 1 to an integrable random variable X, as $n \to \infty$. (Hint: use the strong law of large numbers.)
(b) Suppose specifically that Y_n assumes the values $\frac{1}{2}$ and $\frac{3}{2}$ with probability $\frac{1}{2}$ each. Show that in this case $X = 0$ with probability 1.
(c) Is the martingale (X_n) in (b) uniformly integrable? Why (not)?

3. Consider Brownian motion W_t, defined on some probability space $(\Omega, \mathcal{F}, P)$ and consider the martingale

$$M_a(t) = e^{aW_t - \frac{1}{2}a^2 t},$$

for $a \in \mathbb{R}$. We denote by $\mathcal{F}_s$ the sigma-algebra generated by W_t, $t \leq s$.
(a) Explain why

$$\int_A M_a(s)dP = \int_A M_a(t)dP, \text{ for } \quad s \leq t, \quad A \in \mathcal{F}_s.$$

(b) Differentiate the integral identity in (a) up to four times, and use the result to prove that

$$W^2 - t$$

and

$$W_t^4 - 6tW_t^2 + 3t^2$$

are martingales. (The first of these was already shown to be a martingale in the lectures, by direct computation.)
(c) Show that for every bounded stopping time τ, we have $E(\tau) = E(W_\tau^2)$.

4. Consider a continuous time Markov process X_t on a countable state space I and with Q-matrix $Q = (q_{i,j})$. Let ν be a measure on I.
(a) Show that when $\nu_i q_{i,j} = \nu_j q_{j,i}$ for all $i, j \in I$ $(i \neq j)$, then ν is invariant for X_t.

Consider now the Markov process X_t with state space $\mathbb{Z}$ and Q-matrix given by

$$q_{i,i+1} = \lambda q_i \text{ and } q_{i,i-1} = \mu q_i,$$

and with $q_{i,j} = 0$ for all other i and j. Here, λ and μ are positive constants.
(b) Show that

$$\nu_i = q_i^{-1} \left(\frac{\lambda}{\mu}\right)^i$$

is invariant for X_t.
(c) Show that there is no stationary probability distribution for X_t when the q_i's are constant.
(d) How would the answer to (c) change if the state space would be restricted to the positive integers?
(e) Suppose now that $\lambda = \mu$. For which q_i's are there stationary probability distributions? Motivate your answer.

5. Consider Brownian motion W_t. In this exercise we are interested in those points t of time for which $W_t = 0$. To this end, we assume that W_t is defined on some probability space $(\Omega, \mathcal{F}, P)$, and we write $Z = Z(\omega)$ for the set $\{t \geq 0; W_t(\omega) = 0\}$. Lebesgue measure is denoted by μ.
(a) Show (by interchanging the order of integration) that

$$\int_\Omega \mu(Z(\omega))dP(\omega) = 0,$$

and argue from this that Z a.s. has Lebesgue measure zero.

We are now going to show that each point in $Z(\omega)$ is the limit point of other points of $Z(\omega)$. For this, we use the *strong Markov property* for Brownian motion, a result which we have not proved in the lectures.
(b) Formulate the strong Markov property for Brownian motion. (You are not asked to prove this.)
(c) Show that for every $r \geq 0$,

$$\tau_r(\omega) = \inf\{t; t \geq r, W_t(\omega) = 0\}$$

is a finite stopping time.
(d) Show, using the strong Markov property, that for each r, the same holds a.s. for the first zero of the Brownian motion following r. (You can use the fact, proved in class, that $0 \in Z(\omega)$ is the limit point of other points of $Z(\omega)$.
(e) Show that with probability 1, any point in $Z(\omega)$ is the limit point of other points in $Z(\omega)$.