Exam "Stochastic Processes for Finance"

February 12, 2014

Give crisp and clear computations/argumentations.
Use of notes, textbook(s), calculators (and other electronic equipment) is not allowed.

This exam consists of five exercises of equal weight.
The final grade of this course is based on the average grade for the homework assignments (twenty percent) and the grade for this exam (eighty percent).

1. Consider a three-period binomial model with initial stock price $S_0 = 1$. After each coin toss the stock price is multiplied by $u = 3$ if the outcome of the coin toss is H (which happens with some probability $p \in (0, 1)$), and multiplied by $d = 1/2$ if the outcome is T. Let the interest rate be $r = 1/3$. Suppose that we have a derivative security which pays, at time 3, the amount K if $S_3 = 27$, and the amount 0 otherwise.

(a) Compute the price of the derivative security at time 0.

(b) How much stock is held in the replicating portfolio at times 0 and 1?. (In other words, compute Δ_0, $\Delta_1(H)$ and $\Delta_1(T)$).

2. Let $T > 0$ and consider the following boundary value problem:

$$\frac{\partial f(t,x)}{\partial t} + \frac{1}{2}t^2\frac{\partial^2 f(t,x)}{\partial x^2} = 0, \quad x \in \mathbb{R},\, t \in [0, T],$$
$$f(T, x) = x^3, \quad x \in \mathbb{R}.$$

Use the Feynman-Kac theorem to find an explicit solution, and check this solution.

3. For which value or values (if any at all) of the number a is the process

$$atW^2(t) - \int_0^t (W^2(s) + s)\, ds, \;\; t \geq 0,$$

a martingale?

4. Consider the standard Black-Scholes model for a stock, with parameters $\alpha > 0$ and σ (the volatility). The stock price at time t is denoted by $S(t)$. Suppose that we have a derivative security which pays, at the time of maturity T, the amount K if $S(T) > 2S(T/2)$, and the amount 0 otherwise.
Compute the price at time 0 of this derivative security, for the case where $T = 8$, $\sigma = 1$ and the interest rate r is equal to $3/2$.
Remark: You may express the result in terms of the function $N(x)$, the probability that a random variable with standard normal distribution has value smaller than x.

5. Let a be a real number. Consider the process $R(t), t \geq 0$, defined by

$$R(t) = e^{-2t} + a(1 - e^{-2t}) + 2e^{-2t} \int_0^t e^{2s} dW(s).$$

For which value of a does this process satisfy the following?

$$dR(t) = (1 - 2R(t))\, dt + 2\, dW(t).$$