WRITTEN EXAM: STOCHASTIC PROCESSES FOR FINANCE (CODE X_400352)

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General information. The written exam consists of 7 problems with a varying number of questions in each of them. Provide clear, but brief answers to the questions. Use of books or notes is not allowed.

Grading. The final grade for the course is based on the average grade for homework assignments (50%) and the grade for the written exam (50%). In greater detail, it is computed as follows: each problem of the written exam has an initial weight of 10 points. The number of points for all problems (the maximum is 70) is summed up and multiplied by a factor 1/7 so as to normalise the grade for the written exam to the interval $[0, 10]$. Denoting by x the normalised grade for the written exam and by y the average grade for the homework assignments (also normalised to the interval $[0, 10]$), an average of the two is computed as

$$z = 0.5 \times x + 0.5 \times y.$$

The final grade is obtained by rounding z off according to the faculty regulations.

Problem 1. Provide answers to the following two questions.

(i) Describe the one period binomial model of the stock market.
(ii) Assume that the model in (i) is arbitrage free and compute within it a fair (arbitrage free) price of a European call option with strike K (it is not necessary to show intermediate computations leading to the result).

Problem 2. Let S denote a stock price process and let $K > 0$ be a fixed number. Consider a contract written on S, that is called a *straddle* and that is defined through its payoff

$$\mathcal{X} = \begin{cases} K - S(T) & \text{if } 0 < S(T) \leq K, \\ S(T) - K & \text{if } S(T) > K. \end{cases}$$

Provide answers to the following two questions.

(i) The payoff $\mathcal{X}$ can be expressed in terms of payoffs of appropriately combined European call and put options written on the stock. Show how this can be done.
(ii) Assuming there are no arbitrage possibilities in the market, use (i) to derive a formula for the price of a straddle at time $t = 0$ in terms of the prices of the put and call options.

Problem 3. Consider the boundary value problem

$$\frac{\partial F(t,x)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(t,x)}{\partial x^2} = 0, \quad x \in \mathbb{R}, t \in [0,T),$$
$$F(T,x) = (x-1)^2.$$

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Provide answers to the following two questions.

(i) Use the Feynman-Kac formula to find a solution to the above boundary value problem.

(ii) Check by a direct computation that a solution found through the Feynman-Kac formula indeed solves the above boundary value problem.

Problem 4. Consider a stochastic differential equation

$$dX(t) = \alpha X(t)dt + \sigma dW(t),$$
$$X(0) = 1,$$

and let a stochastic process $\widetilde{X}$ be defined by

$$\widetilde{X}(t) = e^{\alpha t} + \sigma e^{\alpha t} \int_0^t e^{-\alpha s} dW(s).$$

Provide answers to the following two questions.

(i) Show that the process $\widetilde{X}$ satisfies the stochastic differential equation above. Hint: write $\widetilde{X}(t) = Y(t) + Z(t)R(t)$ for

$$Y(t) = e^{\alpha t}, \quad Z(t) = \sigma e^{\alpha t}, \quad R(t) = \int_0^t e^{-\alpha s} dW(s)$$

and next apply a multidimensional version of Itô's formula (with $f(y, z, r) = y + zr$) to $\widetilde{X}$.

(ii) Compute the variance $\mathbb{V}\mathrm{ar}\,[\widetilde{X}(t)]$ of $\widetilde{X}(t)$.

Problem 5. Suppose the Black-Scholes model with parameters $\alpha = 17$, $\sigma = 1$ and $r = 0$ is given, where α is the local mean rate of return of the stock, σ is its volatility and r is the interest rate. Furthermore, let $S(0) = 1$, where $S(t)$ denotes the stock price at time t. An innovative company F&H INC has produced a financial derivative "the Golden Logarithm", which we abbreviate to GL. A holder of GL with maturity time T denoted as $GL(T)$ will, at time T, obtain the sum $\log S(T)$. Note that if $S(T) < 1$, this means that the holder has to pay a positive amount to F&H INC.

Provide an answer to the following question.

(i) Determine a fair (arbitrage free or Black-Scholes) price for $GL(T)$ with maturity $T = 1$.

Problem 6. Suppose the Black-Scholes model is given (with the stock price process denoted by S and with the interest rate denoted by r) and consider a European call option and a European put option, both with the same strike K and the same maturity T. Denote the corresponding fair (Black-Scholes) prices at time t by $C(t)$ and $P(t)$.

Provide answers to the following two questions.

(i) Using a no-arbitrage argument, prove the the put-call parity

$$P(t) = Ke^{-r(T-t)} + C(t) - S(t).$$

Keep your arguments brief.

(ii) Δ of a European call option is given by $\Delta_C = N(d_1)$, where d_1 is some constant and N denotes the standard normal distribution function (this information on d_1 and N is not relevant for this problem). Show that Δ of a European put option with the same strike and maturity is given by

$$\Delta_P = N(d_1) - 1.$$

Problem 7. Provide answers to the following two questions.

(i) Solve the stochastic differential equation

$$dr(t) = -r(t)dt + 2dW(t)$$

for the short rate $r(t)$ in the Vasicek model (remember that the r-dynamics in this stochastic differential equation are considered under the martingale measure $\mathbb{Q}$). Next argue why the distribution of $r(t)$ (here t is fixed) under $\mathbb{Q}$ is normal and why under the real world measure $\mathbb{P}$ there is a positive probability for $r(t)$ becoming negative.

(ii) Give a specification of the Hull-White model of the short rate.