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Give clear, but brief motivations for all your answers.
Calculators are not allowed.
Grade is total credits divided by 10.

1. (20=5+10+5 points.) Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $\mathrm{P}(X_i = 1) = \mathrm{P}(X_i = 0) = 1/2$, for every i. For given $\mu \in \mathbb{R}$, set

$$M_n = \exp\Bigl(\sum_{i=1}^n (X_i + \mu)\Bigr).$$

   a. Show that the filtrations generated by $X_1, X_2, \ldots$ and $M_1, M_2, \ldots$ are the same.
   b. For what value(s) of μ is the sequence $M_1, M_2, \ldots$ a martingale (relative to its own filtration)?
   c. For what value(s) of μ is the sequence $M_1, M_2, \ldots$ a Markov process (relative to its own filtration)?

2. (20=8+5+7 points.) Consider a (discrete time) market that consists of a bank account with fixed interest rate r and a stock with price S_n at time n, for S_n following the binomial tree model:

$$\begin{aligned} S_0 &= 1, \\ \mathrm{P}(S_{n+1} = uS_n \mid S_0, \ldots, S_n) &= p, \qquad n \geq 0, \\ \mathrm{P}(S_{n+1} = dS_n \mid S_0, \ldots, S_n) &= 1 - p, \end{aligned}$$

   where d and u are known numbers satisfying $0 < d < e^r < u$. Consider the option that pays 1 money unit at the fixed time N if the stockprice S_N is strictly below a predescribed level K, and pays 0 otherwise.
   a. Characterize the stock price process in a “risk-neutral” market.
   b. Write S_N in terms of X_N, defined as the number of times $n \in \{0, 1, \ldots, N-1\}$ that $S_{n+1} = uS_n$. What is the distribution of X_N in the “risk-neutral” market?
   c. Give an explicit formula (may be complicated) for the arbitrage-free price of the option.

3. (20=4+7+5+4 points.)
   a. Give the definition of a Brownian motion W.
   b. Show that the stochastic differential equation

$$dX_t = (2X_t + 1)\,dt + 2\sqrt{X_t}\,dW_t, \qquad X_0 = 0.$$

   is solved by $X_t = e^{2t}(\int_0^t e^{-s}\,dW_s)^2$. [Suggestion: first find dY_t^2 for $Y_t = \int_0^t e^{-s} dW_s$.]
   c. Use the Itô isometry to show that $\mathrm{E}X_t = (e^{2t} - 1)/2$.
   d. Show that X_t is distributed as the square of a zero-mean normal variable. [Two or three lines suffice; relate to properties of Brownian motion and the stochastic integral.]

4. (20=4+4+4+4+4 points.) Consider a market that consists of a bank account with zero interest, and an asset with price S_t satisfying

$$dS_t = (2S_t + 1)\,dt + 2\sqrt{S_t}\,dW_t, \qquad S_0 = 1.$$

(Zero interest rate sounds funny, but we can put ourselves in this situation by quoting all prices relative to the bank account. For this problem it just means that the bank account takes the form $B_t \equiv 1$.) Consider an option that pays the amount 1 at time T if $S_T < 1$ and 0 otherwise. Assume that the value of the option can be written as $V_t = G(t, S_t)$, for some smooth function $G\colon [0,T] \times \mathbb{R} \to \mathbb{R}$.

   a. Argue that $\widetilde{W}_t = \int_0^t \big((2S_s + 1)/(2\sqrt{S_s})\big)\,ds + W_t$ is a Brownian motion under the risk-neutral measure.
   b. Use Itô's formula to derive a stochastic differential equation for V_t in terms of partial derivatives of G, and of S_t and $\widetilde{W}_t$.
   c. Use the general formula for option prices as an expectation under the risk-neutral measure to explain why V must be a martingale under the risk-neutral measure.
   d. Use the fact that V is a martingale to derive a partial differential equation for G.
   e. What are the boundary conditions for this equation?

5. (20=5+5+5+5 points.)
   a. Give the definition of the forward rate (at time T contracted at time t).
   b. Give the definition of the short rate.
   c. Give the Hull-White model for the short rate.
   d. The Hull-White model for the term structure can be written as

$$P(t,T) = \mathrm{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s\,ds} \mid \mathcal{F}_t\Big).$$

   Name the entities $P(t,T)$, $\mathbb{Q}$, r_t in this equation. How do $\mathbb{Q}$, r_t and $\mathcal{F}_t$ relate to your answer in c)?