Department of Mathematics	Exam "Stochastic Processes for Finance"
Vrije Universiteit	February 9, 2009

Give your answers in English.
It is not allowed (nor useful) to use calculators.
Good luck!

1. (Arbitrage)

   Let E_t denote the value of one US dollar in euros at time t and $r > 0$ the European (risk free, constant) interest rate. Consider a forward contract corresponding to an agreement to buy one US dollar for K euros at a specified future time T. Use the standard pricing formula ($V_0 = \mathbb{E}_{\mathbb{Q}}(e^{-rT}C)$, where C is the claim and $\mathbb{Q}$ is the martingale measure) to derive the fair value of the strike price K in terms of E_0 and r.

2. (Discrete time process)

   Consider a discrete time process X such that $X_0 = x_0$ and

   $$X_{n+1} = \begin{cases} X_n + a_n + b & \text{with probability } 1/2 \\ X_n - a_n + b & \text{with probability } 1/2 \end{cases}$$

   for some deterministic number b and sequence $(a_n, n \geq 0)$.

   (a) Is X a predictable process? Why/Why not?

   (b) Is X a Markov process? Why/Why not?

   (c) When is X a martingale?

3. (Brownian motion)

   (a) Let X be the process defined by $X_t = aW_{4t}$. For what value of $a > 0$ is X a Brownian motion with respect to its natural filtration?

   (b) Let $Y_t = \sigma W_t + \mu t$ where W_t is a $\mathbb{P}$-BM. Show that there exists a measure $\mathbb{Q}$ such that $Y_t = \tilde{W}_{\sigma^2 t}$ where $\tilde{W}_t$ is a $\mathbb{Q}$-BM. (You don't need to say what $\mathbb{Q}$ is.)

4. (Ito's formula)

   Let W be a Brownian motion and a a real number. Use Ito's formula to show that the following processes are martingales.

   (a) $X_t = \exp\left(aW_t - \frac{a^2 t}{2}\right)$

   (b) $Y_t = tW_t - \int_0^t W_s ds$

5. (Self-financing portfolio)

   Let S and B denote the stock and bond price processes in a Black-Scholes market. Consider a portfolio (φ, ψ), where φ_t is the number of stocks held at time t and ψ_t the number of bonds. Suppose that $\varphi_t = aS_t + b$ and $\psi_0 = 0$. Determine ψ_t in such a way that the portfolio becomes self-financing.

6. (Stock market model)

   Consider a market in which a stock is traded with price process $S_t = \exp\left(W_t - \frac{t}{2}\right)$, where W is a Brownian motion under the "real-world" probability measure $\mathbb{P}$, and with a bank with zero interest. Let $T > 0$ and let $C = f(S_T)$ be a European claim with value V_t at time t.

   (a) What is the martingale measure $\mathbb{Q}$?

   (b) Give an integral expression for the price V_0 at time 0 of the derivative with claim C.

7. (Hull-White model)

   Consider the Hull-White model with

   $$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t,$$

   where W denotes Brownian motion, a and σ are constants and θ is a deterministic function.

   (a) Express r_t as

   $$r_t = e^{-at}[r_0 + \alpha(t) + \sigma Y_t],$$

   where $\alpha(t)$ is given by an ordinary integral and Y_t by a stochastic integral. (Hint: apply Ito's formula to $e^{at}r_t$.)

   (b) Compute the mean and variance of Y_t.

Points:

1:	3	2(a):	2	3(a):	3	4(a):	4	5:	4	6(a):	3	7(a):	3
		2(b):	2	3(b):	2	4(b):	3			6(b):	3	7(b):	2
		2(c):	2										