Department of Mathematics	Exam "Stochastic Processes for Finance"
Vrije Universiteit	February 4, 2008

Give your answers in English.
It is not allowed (nor useful) to use calculators.
Good luck!

1. (Arbitrage arguments)

   Consider a derivative which at a predetermined time T in the future pays to the owner the amount $a(S_T - K)$ if $S_T \geq K$ and $b(K - S_T)$ if $S_T < K$.

   (a) Construct a self-financing portfolio of call and put options which replicates the payoff of the derivative.

   (b) Using the answer of part (a) and an arbitrage argument, express the value of the derivative at time t in terms of the price of certain call and put options.

2. (Discrete-time martingales)

   Show that a discrete time martingale X is predictable if and only if it is constant (i.e., $X_n = X_0$ for all n).

3. (Tree model)

   Let S be a discrete-time process such that

   - $S_0 = s_0 (\neq 0)$
   - for $n > 0$,

   $$S_n = \begin{cases} S_{n-1} & \text{with probability } p_1 \\ uS_{n-1} & \text{with probability } p_2 \\ dS_{n-1} & \text{with probability } p_3 \end{cases}$$

   where $p_1 + p_2 + p_3 = 1$, and u and d are two positive real numbers.

   (a) Give an expression for $\mathbb{E}_p(S_n \mid S_1, \ldots, S_{n-1})$.

   (b) Let $p_1 = 1/2$, $u = 2$, and $d = 1/2$. For what values of p_2 and p_3 is S a martingale (with respect to its natural filtration)?

4. (Brownian motion)

   Let X be the process defined by $X_t = \mu t + \sigma W_t$, where $(W_t, t \geq 0)$ is a $\mathbb{P}$-Brownian motion, and μ and σ are constants. Use Girsanov's theorem and the scaling property of Brownian motion (if W_t is a Brownian motion, then $\frac{1}{\sqrt{\lambda}} W_{\lambda t} \stackrel{d}{=} W_t$) to show that there exists a measure $\mathbb{Q}$ (you don't need to say what it is) such that $X_t = \tilde{W}_{\sigma^2 t}$, where $\tilde{W}$ is a $\mathbb{Q}$-Brownian.

5. (Stochastic calculus)

   Let W be a Brownian motion.

   (a) Compute $\int_0^t W_s \, dW_s$.

   (b) Let S be the process defined by $S_t = \exp(\sigma W_t + \alpha + \beta t)$. Use Itô's formula to obtain a stochastic differential equation for S. For which choice of parameters α, β and σ is S a martingale?

6. (Black-Scholes)

   Let S and B denote the stock and bond price processes in a Black-Scholes market. Assume that the risk-free interest rate is a constant r. Consider a portfolio (φ, ψ), where φ_t is the number of stocks held at time t and ψ_t the number of bonds. Let a, b be two positive real numbers, and suppose that $\psi_0 = 0$ and $\varphi_t = at + b$. Determine ψ_t in such a way that the portfolio becomes self-financing.

7. (Silly stock market model)

   Consider a market in which a stock is traded with price process $S_t = \frac{1}{\sqrt{\lambda}} W_{\lambda t}$, where W is a Brownian motion under the "real-world" probability measure $\mathbb{P}$, and λ is a positive real number. Assume that the risk-free interest rate is zero. Let $T > 0$ and let $C = f(S_T)$ be a European claim with value V_t at time t.

   (a) Explain why the martingale measure $\mathbb{Q}$ is equal to the "real-world" measure $\mathbb{P}$.

   (b) Give an integral expression for the price V_0 of the derivative at time 0.

Points:

1(a):	2	2:	3	3(a):	2	4:	5	5(a):	4	6:	5	7(a):	3
1(b):	1			3(b):	2			5(b):	5			7(b):	4

Grade = (total points+4)/4