Department of Mathematics	Exam "Stochastic Processes for Finance"
Vrije Universiteit	February 5, 2007

Give your answers in English.
It is not allowed (nor useful) to use calculators.
Good luck!

1. (Arbitrage arguments)

   Consider a forward contract corresponding to an agreement to buy an asset on a specified future date T for a specified price K. Assume that at the time the contract is signed the value of the asset is S_0 and let r be the risk-free interest rate. Use an arbitrage argument to obtain the fair price of the forward.

2. (Discrete-time martingales)

   In this exercise time is discrete, $(\mathcal{F}_n)$ is a given filtration. Consider a discrete time process X such that $X_{n+1} = f_n(X_n)$, where f_n are deterministic functions. Show that if X is a martingale, then it is constant, i.e., $X_n = X_0$ almost surely, for all n.

3. (Random walk)

   Let S be the simple random walk. This means that $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, where the X_i's are independent and $\mathbb{P}_p(X_i = 1) = 1 - \mathbb{P}_p(X_i = -1) = p$. Let $(\mathcal{F}_i)$ be the natural filtration of S.

   (a) Explain why S is a Markov process.

   (b) Give an expression for $\mathbb{E}_p(S_{i+1} \mid X_1, \ldots, X_i)$.

   (c) Show that S is a martingale with respect to $(\mathcal{F}_i)$ if and only if $p = 1/2$.

4. (Brownian motion)

   Let W be a Brownian motion and a a positive real number. Denote the natural filtration of W by $(\mathcal{F}_t)$.

   (a) Show that the process X defined by $X_t = a^{-1/2} W_{at}$ is a Brownian motion with respect to its natural filtration.

   (b) Use Ito's formula to show that the process Y defined by $Y_t = W_{at}^2 - at$ is a martingale.

5. (Stochastic calculus)

   Let W be a Brownian motion and denote the natural filtration of W by $(\mathcal{F}_t)$.

   (a) Compute $\int_0^t W_s \, dW_s$.

(b) Compute $\mathbb{E}(\int_0^t W_s\, dW_s)^2$.

(c) Let S be the process defined by $S_t = \exp(\sigma W_t + \alpha + \beta t)$. For which choice of parameters α, β and σ is S a martingale?

6. (Black-Scholes)

Let S and B denote the stock and bond price processes in a Black-Scholes market. Assume that the risk-free interest rate is zero. Consider a portfolio (φ, ψ), where φ_t is the number of stocks held at time t and ψ_t the number of bonds. Suppose that $\varphi_t = t$ and $\psi_0 = 0$. Determine ψ_t in such a way that the portfolio becomes self-financing.

7. (Silly stock market model)

Consider a market in which a stock is traded with price process $S_t = W_t^2 - t$, where W is a Brownian motion under the "real-world" probability measure $\mathbb{P}$, and with a bank with zero interest. Let $T > 0$ and let $C = f(S_T)$ be a European claim with value V_t at time t.

(a) What is the martingale measure $\mathbb{Q}$?

(b) Give an integral expression for the price V_0 of the derivative at time 0.

Norming:

1:	3	2:	3	3(a):	1	4(a):	3	5(a):	5	6:	4	7(a):	4
				3(b):	1	4(b):	2	5(b):	1			7(b):	4
				3(c):	1			5(c):	4				

Grade = (total+4)/4