Department of Mathematics	Exam "Stochastic Processes for Finance"
Vrije Universiteit	December 23, 2005

Give your answers in English.
It is not allowed (nor useful) to use calculators.
Good luck!

1. (Arbitrage arguments)

   Consider a derivative which at a predetermined time T in the future, pays to the owner the amount $|S_T - K|$, where S_T is the value of a particular stock at time T, and K is a fixed number.

   (a) Construct a self-financing portfolio consisting of call options and put options, which replicates the pay-off as this derivative. (Specify the strike prices and expiry times of the options in the portfolio.)

   (b) Using the answer of part (a) and an arbitrage argument, express the value of the derivative at time t in terms of the prices of certain call and put options.

2. (Discrete-time martingales)

   In this exercise time is discrete, $(\mathcal{F}_n)$ is a given filtration. Show that if a (discrete-time) process X is predictable and also a martingale, it is constant, i.e. $X_n = X_0$ almost surely, for all n.

3. (Binomial model)

   Consider a process $S = (S_0, \ldots, S_n)$ which follows an n-period binomial model. The price S_0 at time $t = 0$ is a given number and at each time t the next value S_{t+1} is uS_t or dS_t with probabilities p and $1-p$, respectively, where $p \in (0,1)$ and $d < 1 < u$ are given constants. Let $(\mathcal{F}_t)$ be the natural filtration of S.

   (a) Give an expression for the conditional expectation $\mathbb{E}_p(S_{t+1} \mid \mathcal{F}_t)$.

   (b) Using (a), determine the value of p for which the process S is a martingale with respect to the filtration $(\mathcal{F}_t)$.

4. (Brownian motion)

   Let W be a Brownian motion and a a positive number. Denote the natural filtration of W by $(\mathcal{F}_t)$.

   (a) Show that the process X defined by $X_t = a^{-1/2} W_{at}$ is a Brownian motion with respect to its natural filtration.

   (b) Using Itô's formula, show that the process Y defined by $Y_t = W_t^3 - 3tW_t$ is a martingale.

   (c) Use Itô's formula to obtain a stochastic differential equation for the process Z defined by $Z_t = \exp(aW_t - a^2t/2)$. Deduce that this process is a martingale.

5. (Black-Scholes)

Let B and S be the bond and stock price processes in a Black-Scholes market. These are assumed to satisfy $B_t = \exp(rt)$ and $S_t = \exp(\mu t + \sigma W_t)$, where $r > 0$, μ and σ are given numbers and W is a Brownian motion. Consider a portfolio (φ, ψ), where φ_t is the number of stocks held at time t and ψ_t the number of bonds. Suppose that $\varphi_t = S_t$ and $\psi_0 = 0$. Determine ψ_t in such a way that the portfolio becomes self-financing.

6. (Silly stock market model)

Consider a world in which a stock is traded with price process $S_t = W_t$, where W is a Brownian motion under the real-world probability measure $\mathbb{P}$ and with a bank with zero interest. Let $T \geq 0$ and let $C = f(S_T)$ be a European claim with value V_t at time t.

(a) In this simple model the discounted stock price is already a martingale under $\mathbb{P}$, and hence the martingale measure $\mathbb{Q}$ is equal to $\mathbb{P}$. Use this to give an integral expression for V_0.

(b) Assuming that $V_t = F(t, S_t)$ for some smooth function F, derive a partial differential equation for the pricing function F.

7. (Hull-White)

In the Hull-White model the short rate is assumed to satisfy, under $\mathbb{Q}$, the SDE

$$dr_t = (\theta(t) - ar_t)\,dt + \sigma\,dW_t,$$

where W is a Brownian motion, a, σ are constants and θ is a deterministic function.

(a) Apply Itô's formula to $\exp(at)r_t$ to express $\exp(at)r_t - r_0$ as the sum of a stochastic integral and an ordinary integral.

(b) Using the answer of part (a), determine, for a fixed $t \geq 0$, the distribution of r_t under $\mathbb{Q}$.

Norming:

1(a):	2	2:	3	3(a):	2	4(a):	2	5 :	5	6(a):	3	7(a):	3
1(b):	1			3(b):	1	4(b):	4			6(b):	4	7(b):	2
						4(c):	4						

Grade = (total+4)/4