Exam caput Stochastic Optimization
20 January 2006, duration: 2 hours

This exam consists of **3** problems, each consisting of several questions.
All answers should be motivated, including calculations, formulas used, etc.
The minimal note is 1. All questions are worth 3 points.

Question 1
Consider a semi-Markov process with $\mathcal{X} = \{1, 2, 3\}$. $T(x)$ has a uniform distribution on $[0, x]$ for all $x \in \mathcal{X}$. $p(x, y) = 1/2$ for all $x \neq y$, $x, y \in \mathcal{X}$.

a. List and check for this process the conditions under which the embedded chain has a limiting distribution.
b. Calculate this distribution (π_* in the lecture notes).
c. Use the answer found under b to calculate the time-limiting distribution (ν_* in the lecture notes).

Question 2
Consider a Markov process consisting of 4 states: $\mathcal{X} = \{0, 1, 2, 3\}$. The following transition rates are positive: $\lambda(x, x+1) = 1$ for $x = 0, 1, 2$, and $\lambda(x, x-1) = 1$ for $x = 1, 2, 3$. There are rate costs $c_r(x) = \max\{0, x-1\}$ in state x, $x \in \mathcal{X}$.

a. Formulate the Poisson equations for this system and solve it.

Now we introduce actions, $\mathcal{A} = \{0, 1\}$: $\lambda(x, 0, x+1) = \lambda(x, 1, x) = 1$ for $x = 0, 1, 2$, and $\lambda(x, 0, x-1) = \lambda(x, 1, x-1) = 1$ for $x = 1, 2, 3$. There are rate costs $c_r(x, 0) = c_r(x, 1) = \max\{0, x-1\}$ in state $x \in \mathcal{X}$, and lump costs $c_l(x, 1) = 0.5$ in state $x \in \{0, 1, 2\}$.

b. Formulate the Poisson equations.
c. Solve it using policy iteration starting with the initial policy that takes action 0 in every state. Give the optimal policy.

Question 3
Consider a Markov decision process with Poisson arrivals and N servers, server i having service rate $\mu(i)$, $\mu(1) \geq \cdots \geq \mu(N)$. Every arriving customer has to be assigned to an idle server, when all servers are busy the arriving customer is lost. The objective is to minimize the long-run average number of customers in the system.

a. Formulate the backward recursion equation of this problem.
b. Show with induction to the time parameter of the value function that the policy that assigns to the fastest available server is optimal.