Exam Measure Theory

December 16, 2014, 12.00-14.45

1. Let λ be Lebesgue measure on $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x|$. Describe the measure $\lambda \circ f^{-1}$.

2. Let X be a set and let $(Y, \mathcal{A})$ be a measurable space. Let $T : X \to Y$ be surjective. Finally, $f : X \to \mathbb{R}$ is a function from X to $\mathbb{R}$.
(a) Suppose there exists an $\mathcal{A}/\mathcal{B}$ measurable function $g : Y \to \mathbb{R}$ such that $f = g \circ T$. Show that f is $\sigma(T)/\mathcal{B}$ measurable.

For the remainder of this exercise suppose that $f : X \to \mathbb{R}$ is $\sigma(T)/\mathcal{B}$ measurable.
(b) Suppose that $T(x) = T(x')$. Show that for all $E \in \sigma(T)$ we have $x \in E$ if and only if $x' \in E$.
(c) Show that for x and x' as in (b) we have that $f(x) = f(x')$.
(d) Finally show that there exists an $\mathcal{A}/\mathcal{B}$ measurable function $h : Y \to \mathbb{R}$ such that $f = h \circ T$.

3. (a) Formulate the Dominated Convergence Theorem.
(b) Let, for $n, m = 1, 2, \ldots$, $a_n(m)$ and a_n be real numbers such that $a_n(m) \to a_n$ as $m \to \infty$. Use the Dominated Convergence Theorem to formulate a condition under which $\sum_{n=1}^{\infty} a_n(m) \to \sum_{n=1}^{\infty} a_n$ as $m \to \infty$. Explain your answer.

4. Let f be a non-negative measurable function on a sigma-finite measure space $(X, \mathcal{F}, \mu)$. Let λ denote Lebesgue measure on $\mathbb{R}$. Show that

$$\int_X f d\mu = (\mu \times \lambda)\left(\{(x, y) \in X \times \mathbb{R}; 0 \leq y \leq f(x)\}\right).$$

Do this by first showing this is true when f is an indicator function, then for f a simple function, and finally for f a non-negative function.

5. Let $f_1, f_2, \ldots$ be measurable functions on a sigma-finite measure space $(X, \mathcal{A}, \mu)$. Consider the following theorem: If $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$, then $\sum_{n=1}^{\infty} f_n$ converges almost everywhere and $\int_X \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$. Prove this by using Fubini's theorem on $X \times \{1, 2, \ldots\}$.

6. Let λ be Lebesgue measure on $\mathbb{R}$ and define on $\mathcal{B}$ the function $\mu(A)$ as the number of integers contained in A. Which of the following two statements is (are) true: (1) $\lambda \ll \mu$; (2) $\mu \ll \lambda$. Motivate your answer.