Uitwerkingen 17 april 1992.

Opgave 1

Merk op: ook F^2 $\left\{ \begin{array}{l} \geq 0 \\ \text{stijgend} \\ \text{rechts-continu} \end{array} \right.$ dus we kunnen ook spreken van m_{F^2}

i. $A = (a, b] \subset \mathbb{R}$

$$m_{F^2}((a,b]) = F^2(b) - F^2(a) = \underbrace{(F(b) + F(a))}_{\leq 2M}\underbrace{(F(b) - F(a))}_{m_F(a,b]} \leq 2M\, m_F A$$

ii. Twee geschikte uitwendige maten

bij: premaat bij F, premaat bij F^2

$A \in \mathcal{B}or(\mathbb{R})$

$$m_{F^2}(A) = \inf\left\{ \sum_{i=1}^{\infty} m_{F^2}(a_i, b_i] : A \subset \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$$

$$m^*_{F^2}(A) \leq \inf\left\{ \sum_{i=1}^{\infty} 2M\, m_F(a_i, b_i] : A \subset \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$$

$$= 2M \inf\left\{ \sum_{i=1}^{\infty} m_F(a_i, b_i] : A \subset \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$$

$$= 2M\, m_F A$$

$m^*_F(A)$.

iii. F strikt stijgend oneindig

$\alpha \leq 2M$ (deze doet het ~~xxx~~ al). stel $\alpha < 2M$

$$m_{F^2}((n, n+1]) = (F(n+1) + F(n))(F(n+1) - F(n))$$
$$= (F(n+1) + F(n))\, m_F((n, n+1]) \quad > 0 \text{ elke } n.$$

$$\frac{m_{F^2}((n, n+1])}{m_F((n, n+1])} = F(n+1) + F(n) \leq \alpha \quad [n \in \mathbb{N}]$$

laat $n \to \infty$ dan $2M \leq \alpha$ $\lightning$. Dus de kleinste $\alpha = 2M$.

dus we kunnen gedom. conv. gebruiken. (bijna-overal-versie)

$\int_{B_n^+} (f-f_n)\,d\mu \to 0 \quad [n\to\infty]$

$\boxed{\text{Opgave 3.}}$ $(V,\mathcal{A},\mu)$. $\mu(V)>0$. $(f_n)_{n\in\mathbb{N}}$ $f_n: V\to\bar{\mathbb{R}}$, meetbaar

(i) $f := \liminf f_n = \lim_{m\to\infty} \inf_{n\geq m} f_n = \sup_{m\geq 1} \inf_{n\geq m} f_n$

$h_m := \inf_{n\geq m} f_n$

$\{h_m \geq a\} = \underbrace{\underbrace{\bigcap_{n=1}^{\infty} \{f_n \geq a\}}_{\in \mathcal{A}\ \forall n}}_{\in \mathcal{A}}$

$\{\sup_{m\geq 1} h_m \leq b\} = \underbrace{\bigcap_{m=1}^{\infty} \underbrace{\{h_m \leq b\}}_{\in\mathcal{A},\ \text{elke } m}}_{\in\mathcal{A}}$

(ii) Lemma van Fatou
Is bovendien alle f_n niet-negatief
$\int_V (\liminf f_n)\,d\mu \leq \liminf \int_V f_n\,d\mu$

Bewijs(je): $h_m := \inf_{n\geq m} f_n \quad \geq 0$, meetbaar, stijgend

M.C.V.: $\int_V (\lim_{m\to\infty} \cancel{\inf} h_m)\,d\mu = \lim_{m\to\infty} \int h_m\,d\mu$

$\quad\quad \|$

$\int_V (\liminf f_n)\,d\mu$

$\int h_m\,d\mu \leq \int f_m\,d\mu$ elke m.

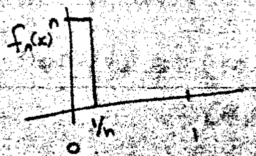
$\lim_{m\to\infty} \int h_m\,d\mu = \liminf_{m\to\infty} \int h_m\,d\mu \leq \liminf_{m\to\infty} \int_V f_m$

$\boxed{\text{Opgave 2.}}$ $(V,\mathcal{A},\mu)$

$f, f_1, f_2, \ldots \in \mathcal{E}\mathcal{A}: V\to\bar{\mathbb{R}} \geq 0$.

(i) zie diktaat.

(ii) $([0,1], \mathcal{B}or[0,1], m)$ m: Borelmaat op $[0,1]$



$f_n := n\,1_{[0,\frac{1}{n}]}$ $[n\in\mathbb{N}]$

$f :\equiv 0$

$f_n \to f$ m-b.o. (niet voor $x=0$ maar $m(\{0\})=0$)

$\int_{[0,1]} |f_n - f|\,m(dx) = n\cdot\frac{1}{n} \equiv 1 \not\to 0$

(iii) Onderstel bovendien $\int_V f\,d\mu = \int_V f_n\,d\mu = 1$, elke n

$\int_V |f-f_n|\,d\mu = \int_{B_n^+} (f-f_n)\,d\mu + \int_{B_n^-} (f-f_n)\,d\mu$

MAAR

$\int_{B_n^+} (f-f_n)\,d\mu + \int_{B_n^-} (f-f_n)\,d\mu = \int_V (f-f_n)\,d\mu = 0$ hier gebruik je gegeven $\forall n$

dus $-\int_{B_n^-} (f-f_n)\,d\mu = \int_{B_n^+} (f-f_n)\,d\mu$

dan $\int_V |f-f_n|\,d\mu = 2\int_{B_n^+} (f-f_n)\,d\mu$.

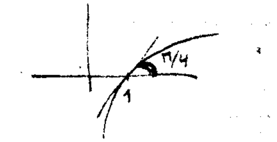
(iv) $\int_{B_n^+} (f-f_n)\,d\mu = \int_V (f-f_n)1_{B_n^+}\,d\mu$

- $(f-f_n)1_{B_n^+} \geq 0$ en $(f_1-f_n)1_{B_n^+} \leq f$ $\forall n$
- $\int_V f\,d\mu < \infty$ $(=1)$.
- $(f-f_n)1_{B_n^+} \to 0$ μ-b.o.; als $f_n \to f$ μ-b.o.

Immers $|(f_1-f_n)1_{B_n^+}| \leq |f-f_n|$

(iii). $\varphi : V \to \mathbb{R}$, $\underset{\Downarrow}{\underline{\text{positief}}}$, meetbaar

$\varphi(x) > 0 \ \forall x$.



$0 < \alpha < 1$

$$\liminf_{n\to\infty} \int_V \underbrace{n \log\{1+(\varphi/n)^\alpha\}}_{\geq 0 \text{ en }} d\mu \qquad \alpha>0 \quad \left(\frac{\varphi(x)}{n}\right)^\alpha \to 0 \ [n\to\infty]$$

$$\underbrace{\frac{\log\{1+(\varphi/n)^\alpha\}}{(\varphi/n)^\alpha}}_{\to 1 \ [n\to\infty]} \cdot \underbrace{(\varphi/n)^\alpha \cdot n}_{= \varphi^\alpha n^{1-\alpha}}$$

$\alpha < 1$: $\to \infty \ [n \to$

(elke $x \in V$.

Conclusie :

$\liminf_{n\to\infty} n \log\{1+(\varphi/n)^\alpha\} \equiv \infty \quad [\forall x \in V]$

We kunnen Fatou gebruiken.

$$\liminf \int_V n \log\{1+(\varphi/n)^\alpha\} d\mu \geq \int_V \infty \, d\mu = \infty \quad (\text{als } \mu(V) > 0$$

$\boxed{\text{Opgave 4}}$ $\quad m_2 = m \otimes m$

$0 < a < b \qquad I := \mathcal{R}\text{-}\int_0^\infty (e^{-ay} - e^{-by}) \frac{1}{y} \, dy$.

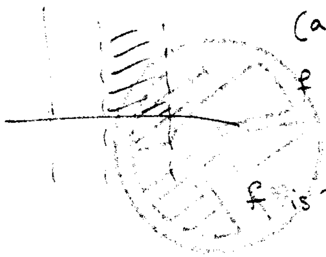
(ii). $\lim_{y \downarrow 0} \dfrac{e^{-ay} - e^{-by}}{y} = \lim_{y\downarrow 0} \dfrac{(e^{-ay}-1)-(e^{-by}-1)}{y} = -a-(-b)$
$= b-a$.

$$- \int_1^\infty \underbrace{\frac{e^{-ay} - e^{-by}}{y}}_{\geq 0 \text{ en }} dy$$

$\leq e^{-ay} - e^{-by}$ en $\mathcal{R}\text{-}\int_1^\infty (e^{-ay} - e^{-by}) \, dy < \infty$.

dus is convergent.

(ii.) $f(x,y) := \begin{cases} e^{-xy}, & (x,y) \in (a,b) \times \mathbb{R}_+ \\ 0, & \text{elders.} \end{cases}$



$(a,b) \times \mathbb{R}_+ \in Bor(\mathbb{R}^2)$.

f is continu op $(a,b) \times \mathbb{R}_+$

" $\quad ((a,b) \times \mathbb{R}_+)^c$.

f is Borelmeetbaar

(i). $I = \int_{(a,b)} \left(\int_{\mathbb{R}^+} f(x,y) \, m(dy) \right) m(dx)$.

$$= \int_{(a,b)} \Big(\underbrace{\int_{\mathbb{R}_+} e^{-xy} m(dy)}_{\overset{?}{=} \int_0^\infty e^{-xy} dy \;=\; -\frac{1}{x} e^{-xy}\big|_0^\infty \;=\; \frac{1}{x}} \Big) m(dx).$$

$= \int_{(a,b)} \frac{1}{x} m dx \overset{?}{=} \mathcal{R}\text{-}\int_a^b \frac{1}{x} dx = \log\left(\frac{b}{a}\right)$

Anderzijds :

$$\int_{\mathbb{R}_+} \left(\int_{(a,b)} f(x,y) \, m dx \right) m dy = \int_{\mathbb{R}_+} \Big(\underbrace{\int_{(a,b)} e^{-xy} m(dx)}_{\mathcal{R}\text{-}\int_a^b e^{-xy} dx \;=\; -\frac{1}{y} e^{-xy}\big|_a^b} \Big) m(dy)$$

$= \frac{1}{y}(e^{-ay} - e^{-by})$ $\left(:= b-a \text{ als } y=0 \right)$

$$= \int_{\mathbb{R}_+} \frac{1}{y} (e^{-ay} - e^{-by}) \, m(dy) \overset{?}{=} \mathcal{R}\text{-}\int_0^\infty \frac{1}{y} (e^{-ay} - e^{-by}) \, dy = I.$$

dus $I = \log \frac{b}{a}$.