Exam Asymptotic Statistics, Mathematische Statistiek

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1. a) Consider a sequence of random variables (X_n) for which $E|X_n| = O(1)$ as $n \to \infty$. Show that this sequence is tight. Can tightness also be proved under the assumption that $EX_n = O(1)$ as $n \to \infty$?

   b) Suppose that $\sqrt{n}(T_n - \theta)$ converges in distribution. Show that $T_n \to^P \theta$ as $n \to \infty$.

   c) Consider two sequences of random variables (X_n) and (Y_n) such that $X_n \rightsquigarrow N_1$ and $Y_n \rightsquigarrow N_2$ for two independent standard normal random variables N_1 and N_2. Prove or disprove (by counterexample) that this implies

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}.$$

   d) Consider an i.i.d. sequence of random variables $X_1, X_2, \ldots$, where X_i is uniformly distributed on the interval $(0, \theta)$ for some $\theta > 0$. Show that $\max_{1 \le i \le n} X_i \to^P \theta$ as $n \to \infty$.

2. Consider the random vector $Y_n = (Y_{n,1}, \ldots, Y_{n,k})$, multinomially distributed with parameters n and $(p_1, \ldots, p_k)$

   a) Formulate a theorem for the asymptotic distribution (for $n \to \infty$) of the sequence of random variables X_n, where

$$X_n = \sum_{i=1}^{k} \frac{(Y_{n,i} - np_i)^2}{np_i}$$

   b) Prove this theorem.

3. The random variables $X_1, X_2, \ldots$ are independent and distributed according to the Poisson distribution with (unknown) parameter $\theta > 0$:

$$P_\theta(X_1 = k) = \frac{\theta^k}{k!} e^{-\theta}, \quad k = 0, 1, 2, \ldots$$

   This means that the expectation as well as the variance of X_1 is θ. Denote by $\bar{X}_n$ the mean of the first n X_i's in the sequence.

   a) Show that $\bar{X}_n$ is a consistent estimator of θ and derive the asymptotic distribution of $\sqrt{n}(\bar{X}_n - \theta)$.

   b) Construct a variance stabilizing transformation ϕ for the sample mean in this model and derive the asymptotic distribution of $\sqrt{n}(\phi(\bar{X}_n) - \phi(\theta))$.

4. Consider the location family of normal distributions with unit variance:

$$p_\theta(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\theta)^2/2} \quad x \epsilon \mathbb{R}.$$

   Let $X_1, X_2, \ldots$ be independent and normally distributed random variables with density p_{θ_0}, and 'true parameter' θ_0. Define, for $\theta \epsilon \mathbb{R}$, the function

$$\psi_\theta(x) = (x - \theta)^3 \quad (x \epsilon \mathbb{R})$$

and consider the Z-estimator $\hat{\theta}_n$ defined as a zero of the function

$$\Psi_n(\theta) = \mathbb{P}_n \psi_\theta = \frac{1}{n} \sum_{i=1}^{n} \psi_\theta(X_i)$$

a) Using properties of the function Ψ_n, show that $\hat{\theta}_n$ is well defined. In other words: show that Ψ_n has exactly one point where it becomes zero.

b) Prove that $\hat{\theta}_n \to^P \mu$.

You may use that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^6 e^{-x^2/2}\, dx = 15$.

c) The random variables $\sqrt{n}(\hat{\theta}_n - \theta_0)$ are asymptotically normally distributed. Use the parameter you expect for the asymptotic variance to compute the asymptotic relative efficiency of $\hat{\theta}_n$ and the sample mean $\bar{X}_n$

5. a) Given a sample $X_1, \ldots, X_n$ from a distribution with probability density f, give the definition of the kernel estimator $\hat{f}_{n,h}$ based on this sample.

b) Show that the Mean Integrated Squared Error of $\hat{f}_{n,h}$ decomposes in a bias- and variance term.

c) Formulate the Glivenko Cantelli theorem.

Grading:

1a: 2	1c: 2	2a: 1	3a: 2	4a: 2	4c: 3	5b: 3
1b: 2	1d: 3	2b: 4	3b: 3	4b: 3	5a: 3	5c: 3

The final grade is computed as follows: $\frac{\text{number of points} + 4}{4}$. Good luck with the exam!