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Vrije Universiteit	21 December 2004

Exam Mathematische Statistiek (6 ECTS): problems 1, 2, 3, 4, 5.
Exam Asymptotic Statistics (8 ECTS): problems 1, 2, 3, 4, 6.
Give clear but brief explications of your answers!
You may write your answers in Dutch, English or French.

1. The random variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are independent, with X_i Poisson distributed with parameter θ and Y_i Poisson distributed with parameter η ($i = 1, \ldots, n$). (The Poisson distribution with parameter θ puts probability $e^{-\theta}\theta^k/k!$ at the points $k \in \{0, 1, 2 \ldots\}$ and has mean and variance θ.)
   a. Determine constants a_n and b_n such that the sequence $a_n(\bar{X}_n/\bar{Y}_n - b_n)$ converges in distribution to a normal distribution with a positive variance. Which variance?
   b. Derive an asymptotic confidence interval for θ/η.

2. Assume that $(Y_1, X_1), \ldots, (Y_n, X_n)$ are independent and identically distributed stochastic vectors distributed according to the non-linear regression model

$$Y_i = \cos(\theta_0 X_i) + e_i,$$

for $\theta_0 \in (0, \infty)$ and independent random variables X_i and e_i with the standard exponential distribution on $(0, \infty)$ and a distribution with $\mathrm{E}e_i = 0$ and $\mathrm{E}e_i^2 < \infty$, respectively ($i = 1, 2, \ldots, n$). Let $\hat{\theta}_n$ be the point of minimum of

$$\theta \mapsto \sum_{i=1}^{n} \bigl(Y_i - \cos(\theta X_i)\bigr)^2, \qquad \theta \geq 0.$$

   a. Give a heuristic argument showing that the sequence $\hat{\theta}_n$ converges in probability to θ_0.
   b. Which limit distribution do you expect for the sequence $\sqrt{n}(\hat{\theta}_n - \theta_0)$?
   c. Assume that each e_i possesses the standard normal distribution. Find an expression for the Fisher information for θ.
   d. Relate the answers to questions b) and c) by reference to a theorem on the asymptotic properties of certain M-estimators.

3. Assume that X_n possesses a multinomial distribution with parameters n and $p = (p_1, \ldots, p_k)$.
   a. Formulate a theorem concerning the limit in distribution of the sequence of variables $\sum_{i=1}^k (X_{ni} - np_i)^2/(np_i)$ as $n \to \infty$.
   b. Prove this theorem.

4. Let X and Y be independent standard normal variables.
   a. Show that the variables $X - Y$ and $X + Y$ are independent.
   b. Show that the variables $\frac{1}{2}(X^2 - Y^2)$ and XY are identically distributed.

5. *(Only Mathematische Statistiek.)* Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with density f.
   a. Give the formula for a kernel estimator $\hat{f}_n$ for f.
   b. Express the bias of $\hat{f}_n(x)$ in f (and the kernel).
   c. Give the definition of the mean integrated square error (MISE) of $\hat{f}_n$.

6. *(Only Asymptotic Statistics.)* Consider the following statistical experiments (i) and (ii):

(i) We observe an i.i.d. sample $X_1, \ldots, X_n$ from a density $x \mapsto p_\theta(x)$ for which the corresponding model $\{p_\theta : \theta \in \mathbb{R}\}$ satisfies the local asymptotic normality assumption: for any $h \in \mathbb{R}$:

$$\log \prod_{i=1}^{n} \frac{p_{\theta+h/\sqrt{n}}(X_i)}{p_\theta(X_i)} = h\Delta_{n,\theta} - \tfrac{1}{2}h^2 I_\theta + o_P(1),$$

under θ, where $I_\theta \in (0, \infty)$ and $\Delta_{n,\theta}$ is a sequence of random variables that tends in distribution under θ to the normal distribution with mean zero and variance I_θ.

(ii) We observe a single observation X from the $N(h, 1/I_\theta)$-distribution, where h is the unknown parameter.

a. Formulate a theorem that relates these two experiments.

b. Suppose that $S_n \rightsquigarrow S$ implies that $\liminf \mathrm{E}S_n^2 \geq \mathrm{E}S^2$, for any sequence of random variables S_n and S.

c. It is known that in experiment (ii) any (randomized) estimator T of h satisfies $\sup_{h \in \mathbb{R}} \mathrm{E}_h(T - h)^2 \geq 1/I_\theta$. What does this imply for

$$\sup_{h \in \mathbb{R}} \mathrm{E}_{\theta+h/\sqrt{n}} n(T_n - \theta - h/\sqrt{n})^2$$

for estimators T_n in the experiments (i), as $n \to \infty$? (You may (but do not have to) assume that the sequence $\sqrt{n}(T_n - \theta)$ tends in distribution to a limit.)

Credits:

1a: 6	2a: 5	3a: 4	4a: 3	5a: 3	6a: 6
1b: 4	2b: 5	3b: 5	4b: 2	5b: 4	6b: 1
	2c: 5			5c: 3	6c: 3
	2d: 5				

Mark = total/6+1.