

This exam has 5 pages and 8 exercises. The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

1. Sets and binary relations (*4 + 4 + 3 + 3 + 3 points*)

- (a) Construct a Venn diagram for each of the following three formulas. Clearly denote how the construction is obtained and which area is given by the formula.

$$(A \cap B) \cup (A \cap C) \cup (B \cap C), \quad (A' \setminus C) \cap B', \quad (B \cup C)' \cap A'.$$

- (b) Prove the equality of the following formula with the laws of algebra for sets.

$$(A' \setminus C)' \cap B = (A \cap B) \cup (B \cap C).$$

- (c) Give the inverse relations of the following binary relations.

- i. *HasTheSameGradeAs*,
- ii. *IsSubsetOf*,
- iii. *IsGrandparentOf*,
- iv. *IsMotherOf*,
- v. *IsElementOfTheEquivalenceClassOf*.

- (d) Simplify the following relations in the composition.

- i. *IsChildOf* $\circ$ *IsSiblingOf*,
- ii. *IsChildOf* $\circ$ *IsMarriedTo*.

- (e) Check if the following relations are reflexive and transitive. Motivate your answer.

- i. *IsGreaterThan*,
- ii. *IsNotADivisorOf*,
- iii. *DiffersByAtLeastTwoDaysFrom*,
- iv. *HasADivisorInCommonWith*.

2. **Relations** ($4 + 2 + 2 + 4 + 2$ points)

We are given the following set of numbers

$$Div24 := \{1, 2, 3, 4, 6, 8, 12, 24\}$$

(the set of all divisors of 24), with the binary relation *Divides* in *Div24* defined by “ x is a divisor of y ”. It is well known that this relation induces a partial order on the set (one does not need to show this).

- (a) Represent the relation *Divides* in *Div24* by a Hasse diagram. While constructing the Hasse diagram, please list the sets Gx as well.
- (b) Now take subset $A := \{2, 4, 6, 12\}$ of *Div24*. Does A have a smallest element according to the order relation *Divides*? If so, please give this element; if not, please list all minimal elements. Clearly mention if these do not exist either.
- (c) (Follow up on part b) Does A have a largest element according to the order relation *Divides*? If so, please give this element; if not, please list all maximal elements. Clearly mention if these do not exist either.
- (d) Consider the equivalence relation R in the set *Div24*, that is defined by $w_1 R w_2$ if and only if the text representation of w_1 and w_2 end with the same letter, e.g., $1 R 3$ since the text representation of $1 = \text{one}$ and $3 = \text{three}$ both end with ‘e’. How many different equivalence classes are there?
- (e) Give a full system of representatives for the equivalence relation R .

3. **Syntax** ($3 + 3$ points)

- (a) Draw the parse tree of the formula $\neg p \wedge q \rightarrow \neg(r \vee p)$
- (b) Compute in the parse tree bottom-up the truth value of this formula, given the truth values T, F, T , for p, q, r , respectively.

4. **Logic** ($4 + 2 + 3 + 3$ points)

- (a) Investigate the validity of the semantic entailment

$$(p \rightarrow q) \rightarrow p \models p$$

(Show clearly how you get to your answer.)

- (b) What does it mean to say: “the set of formulas $\{\phi_1, \phi_2\}$ ” is satisfiable?
- (c) Is the set $\{(p \rightarrow q) \rightarrow p, \neg p\}$ satisfiable?
(Motivate your answer.)
- (d) Give a CNF for the formula $(p \rightarrow q) \rightarrow p$

5. **Functional completeness** (3 + 3 points)

- (a) What does it mean to say that a system of connectives is *functional complete* (or *adequate*)?
- (b) Is the system of connectives $\{\vee, \rightarrow\}$ adequate?
Give a short motivation of your answer.

6. **Binary arithmetic** (2 + 3 points)

- (a) Write 15 as binary number.
- (b) What is the result of the following binary addition?

$$\begin{array}{r} 1 \ 0 \ 1 \ 1 \ 1 \\ 1 \ 0 \ 1 \ 1 \ + \\ \hline \end{array}$$

7. **Boolean functions** (4 + 3 points)

Given is the boolean function $f(x, y)$, having value 1 if at least one of the two variables x, y is 0, and value 0 otherwise.

- (a) Draw a logic circuit with $\wedge$ -, $\vee$ - and $\neg$ -gates for $f(x, y)$.
- (b) Draw a reduced BDD for the function $f(x, y)$

8. **Predicate logic** (3 + 3 + 3 points)

Translate the following sentences to predicate logic using the specification:

Wx: x works
Bxy: x is brother of y
Hx: x is at home
a: Anna

- (a) Some of Anna's brothers are at home
- (b) Anna works, but her brothers do not work
- (c) If all her brothers are at home, then Anna does not work

9. **Functions** (2 + 4 points)

- (a) What is the difference between the image and the range of a typed function $f : A \rightarrow B$.
- (b) Let $A := \{a, b, c, d, e\}$ and $V := A \times A$. Given is a function $value : V \rightarrow \{1, \dots, 10\}$ of which the value is determined by the sum of the letter values ($a = 1, b = 2, c = 3, d = 4, e = 5$), e.g., $value(ab) = 1 + 2 = 3$. Is this function total? Is this function surjective? Please provide arguments.

10. **Induction and Recursion** (4 + 4 points)

- (a) Consider a sequence of real-valued numbers $(t_n)_{n=1}^{\infty}$ defined recursively by

$$t_1 := 1, \quad t_{n+1} := t_n + (3n - 2).$$

- i. Calculate the terms $t_2, \dots, t_6$ of this sequence.
- ii. Prove by mathematical induction that

$$t_n = \frac{n(3n - 1)}{2}, \quad n \geq 1.$$

- (b) In the set $V := \{1, 2, \dots, 6\}$ we have a binary relation R defined by

$$R := \{< 1, 2 >, < 2, 3 >, < 3, 1 >, < 4, 5 >, < 5, 6 >, < 6, 4 >\}.$$

- i. Depict the relation R as a directed graph as well as $R \circ R$.
- ii. Describe or depict the transitive closure of R .

Algebra for sets

Commutativity:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Idempotence:

$$A \cup A = A$$

$$A \cap A = A$$

Associativity:

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Complement:

$$A \cup A' = U$$

$$A \cap A' = \emptyset$$

Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

DeMorgan's Laws:

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

Identities:

$$A \cup U = U \text{ en } A \cup \emptyset = A$$

$$A \cap U = A \text{ en } A \cap \emptyset = \emptyset$$

Involution:

$$(A')' = A$$