Uitwerking tentamen Lineaire Algebra 22-8-2001

1. a)
$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & b_1 \\ 1 & -1 & 0 & b_2 \\ -1 & -1 & -1 & b_3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & b_2 \\ 0 & 2 & 1 & b_1 \\ 0 & -2 & -1 & b_3+b_2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & b_2 \\ 0 & 2 & 1 & b_1 \\ 0 & 0 & 0 & b_1+b_2+b_3 \end{array}\right)$$

precies dan consistent als $\underline{b_1+b_2+b_3=0}$

b) Uit a) volgt: rang$(A) = 2$, want er zijn 2 pivots. Dus dim Nul$(A) = n - r = 3 - 2 = 1$. Een basis voor Nul(A) is bijv.

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right\}$$

c) Los op: $A^T A \hat{\underline{x}} = A^T \underline{y}$ of wel

$$\begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix} \hat{\underline{x}} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 6 & 3 \\ 1 & 3 & 2 \end{pmatrix} \hat{\underline{x}} = A^T \underline{y} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

Welnu:

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 6 & 3 & 2 \\ 1 & 3 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 6 & 3 & 2 \\ 0 & -6 & -3 & -2 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 6 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 6 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 is vrij
$x_1 = -\frac{1}{2}x_3$
$x_2 = \frac{1}{3} - \frac{1}{2}x_3$

Dus alg. kleinste kwadraten-oplossing: $\hat{\underline{x}} = \begin{pmatrix} 0 \\ 1/3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$

d) In elk geval is $\lambda = 0$ een e.w. met e.r.: Nul(A). Op zoek naar evt. andere e.w. van A:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 & 1 \\ 1 & -1-\lambda & 0 \\ -1 & -1 & -1-\lambda \end{vmatrix} = 1(-1-1-\lambda) + (-1-\lambda)(-\lambda(-1-\lambda) - 2) =$$

$$= -2-\lambda + (-1-\lambda)(\lambda+\lambda^2-2) = -2-\lambda-\lambda-\lambda^2+2+$$
$$-\lambda^2-\lambda^3+2\lambda = -\lambda^3-2\lambda^2 = 0 \Leftrightarrow \lambda=0\ (2\times) \vee \lambda=-2$$

Eigenruimte bij $\lambda=-2$:

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

basis bijv. $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

A is niet diagonaliseerbaar, want de alg. multipl. v/d e.w. $\lambda=0$ is 2 maar de meetk. mult. is 1
(Er is dus geen basis van e.v. te vinden)

2. a) $\det B = \begin{vmatrix} 3 & 6 & -1 \\ 0 & 2 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 3(-2-1)+1(6+2) = -1$

$\det(B^T B) = \det B^T * \det B = \det B * \det B = +1$

b) $x_3 = \dfrac{\begin{vmatrix} 3 & 6 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix}}{\det B} = \dfrac{-2}{-1} = +2$

c) $$\left(\begin{array}{ccc|ccc} 3 & 6 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & -3 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 & -3/2 & -3 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -3 & -5 \\ 0 & 1 & 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 2 & -3 & -6 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5 & -8 \\ 0 & 1 & 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 2 & -3 & -6 \end{array}\right)$$ Dus $B^{-1} = \begin{pmatrix} 3 & -5 & -8 \\ -1 & 2 & 3 \\ 2 & -3 & -6 \end{pmatrix}$

Controleren is nuttig!

3. a) Bij $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ en $\underline{v}_3 = \begin{pmatrix} 4 \\ 1 \\ 6 \\ -1 \end{pmatrix}$

dan $\underline{v}_1 \perp \underline{v}_2$ (gelukkig maar). Neem dus $\underline{u}_1 = \underline{v}_1$

$\underline{u}_2 = \underline{v}_2$ en $\underline{u}_3 = \underline{v}_3 - \dfrac{\underline{v}_3 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 - \dfrac{\underline{v}_3 \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2 =$

$$= \begin{pmatrix} 4 \\ 1 \\ 6 \\ -1 \end{pmatrix} - \frac{18}{9} \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix} - \frac{9}{9} \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 6 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 4 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix} =$$

$= \begin{pmatrix} 0 \\ -1 \\ 2 \\ 2 \end{pmatrix}$. Neem dus als <u>orthonormale</u> basis: $\left\{ \frac{1}{3}\begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 0 \\ -1 \\ 2 \\ 2 \end{pmatrix} \right\}$

b) Als $QR = C$ dan $Q^T Q R = R = Q^T C =$

$$= \frac{1}{3} \begin{pmatrix} 1 & 0 & 2 & -2 \\ 2 & 2 & 0 & 1 \\ 0 & -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 1 \\ 2 & 0 & 6 \\ -2 & 1 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 9 & 0 & 18 \\ 0 & 9 & 9 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 6 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

c) $\hat{\underline{x}} = \dfrac{\underline{x} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 + \dfrac{\underline{x} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} \underline{u}_2 + \dfrac{\underline{x} \cdot \underline{u}_3}{\underline{u}_3 \cdot \underline{u}_3} \underline{u}_3 = \dfrac{1}{9} \underline{u}_1 + \dfrac{2}{9} \underline{u}_2 + \underline{0}$

$= \frac{1}{9} \begin{pmatrix} 5 \\ 4 \\ 2 \\ 0 \end{pmatrix}$.

4. a) $T(\lambda p + \mu q) = \begin{pmatrix} (\lambda p + \mu q)(0) \\ (\lambda p + \mu q)(1) \\ (\lambda p + \mu q)(2) \end{pmatrix} = \begin{pmatrix} \lambda p(0) + \mu q(0) \\ \lambda p(1) + \mu q(1) \\ \lambda p(2) + \mu q(2) \end{pmatrix} =$

$= \lambda \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix} + \mu \begin{pmatrix} q(0) \\ q(1) \\ q(2) \end{pmatrix} = \lambda T(p) + \mu T(q)$.

b) Stel $T(p) = \underline{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, dan $p(0) = p(1) = p(2) = 0$
maar het enige polynoom $p \in \mathbb{P}_2$ met tenminste 3 nulpt^n is het 0-polynoom. <u>Dus T is 1-1</u>

c) T is een <u>lineaire</u> afb van <u>3</u>-dim $\mathbb{P}_2$ naar <u>3</u>-dim $\mathbb{R}^3$. Bovendien is T 1-1. <u>Dus is T óp</u>
(Stelling)

5. a) <u>Onwaar</u> : Neem bijv $A = (0\ 0)$ dan $A\underline{x} = 1$ strijdig

b) <u>Onwaar</u> : Neem $A = B = C = D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ dan A inverteerbaar,

maar $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ niet invert. (gelijke rijen!)

c) <u>Onwaar</u> : Neem A een 4×6 0-matrix dan

$\dim \mathrm{Nul}(A) = n - r = 6 - 0 = 6$

d) <u>Waar</u> : Stel $B = PAP^{-1}$ dan $B^k = (PAP^{-1})^k = PA^kP^{-1} =$

$= PIP^{-1} = PP^{-1} = I$

e) <u>Waar</u> : $Q(\underline{u}) = \underline{u}^T A \underline{u}$ met $A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ en A heeft

2 pos. eigenw., nl $3/2$ en $1/2$.

f) <u>Waar</u> : $\det(A^k) = (\det A)^k = \det I = 1$

dus $\det A \neq 0$ dus A invert.

6. a)(i) $S_{2\times2}$ niet-leeg want de 2×2 nul-matrix is symm. en zit dus in $S_{2\times2}$

(ii) de som van symm. 2×2 matrices is weer symm.

$((A+B)^T = A^T + B^T = A + B)$

(iii) als A symm. dan λA ook weer symm.

b) (i) Stel $\underline{x} = \underline{x}_1 + \underline{c}$ met $\underline{c} \in \mathrm{Nul}(A)$ dan

$A\underline{x} = A(\underline{x}_1 + \underline{c}) = A\underline{x}_1 + A\underline{c} = \underline{b} + \underline{0} = \underline{b}$

(ii) Stel $A\underline{x} = \underline{b}$ dan $A\underline{x} = A\underline{x}_1$, dus

$A\underline{x} - A\underline{x}_1 = A(\underline{x} - \underline{x}_1) = \underline{0}$ dus $\underline{x} - \underline{x}_1 = \underline{c}$

voor zekere $\underline{c} \in \mathrm{Nul}(A)$

Dus $\underline{x} = \underline{x}_1 + \underline{c}$ met $\underline{c} \in \mathrm{Nul}(A)$