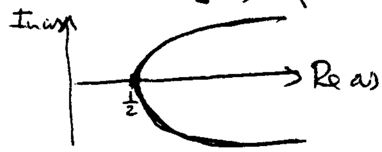
Uitwerking tentamen Wiskunde I   19-10-1999

① a) $|z| = \sqrt{1^2+(-\sqrt{3})^2} = \sqrt{4} = 2 \qquad \arg(z) = -\pi/3$

b) $\dfrac{2+3i}{4+i} = \dfrac{2+3i}{4+i} \cdot \dfrac{4-i}{4-i} = \dfrac{11+10i}{17} = \dfrac{11}{17} + i\,\dfrac{10}{17}$. Dus Re deel $= \frac{11}{17}$, Im deel $= \frac{10}{17}$

c) Stel $w = z+i$. Dan $w^3 = -i = e^{-\frac{\pi}{2}i}$. Dus $w_1 = e^{-\frac{\pi}{6}i} = \frac{1}{2}\sqrt{3} - \frac{1}{2}i$,
$w_2 = e^{+\frac{\pi}{2}i} = +i$ en $w_3 = e^{-\frac{5}{6}\pi i} = -\frac{1}{2}\sqrt{3} - \frac{1}{2}i$. Dus $z_1 = \frac{1}{2}\sqrt{3} - \frac{3}{2}i$,
$z_2 = 0$ en $z_3 = -\frac{1}{2}\sqrt{3} - \frac{3}{2}i$.

d) Stel $z = x+iy$. Dan $\mathrm{Re}(z) = x$ en $|z-1| = \sqrt{(x-1)^2+y^2}$.
Dus $x = \sqrt{(x-1)^2+y^2}$ levert $x^2 = x^2-2x+1+y^2$, oftewel $y^2 = 2x-1$
Dit is een (horizontale) parabool.



② a)
$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ -1 & 3 & -2 & 0 & 1 & 0 \\ -2 & 1 & -8 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & 2 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 5 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -14 & -9 & -3 \\ 0 & 1 & 0 & -4 & -2 & -1 \\ 0 & 0 & 1 & 5 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -22 & -13 & -5 \\ 0 & 1 & 0 & -4 & -2 & -1 \\ 0 & 0 & 1 & 5 & 3 & 1 \end{array}\right)$$

A^{-1} ↑

b) m.b.v. A^{-1} wordt dit: $\underline{x} = A^{-1}\underline{b} = \begin{pmatrix} -22 & -13 & -5 \\ -4 & -2 & -1 \\ 5 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -28 \\ -4 \\ 7 \end{pmatrix}$

③ a) $B \cdot \underline{x} = \begin{pmatrix} 2 & 0 & -1 \\ 3 & -3 & 3 \\ -2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \lambda \cdot \underline{x}$

Dus $\underline{x}$ is een eigenvector van B bij eigenwaarde $\lambda = 0$.

b) $\det(B-\lambda I) = \begin{vmatrix} 2-\lambda & 0 & -1 \\ 3 & -3-\lambda & 3 \\ -2 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -3-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 3 & -3-\lambda \\ -2 & 0 \end{vmatrix} =$

$$= (2-\lambda)(3+\lambda)(\lambda-1) + 2(3+\lambda) = (3+\lambda)(-\lambda^2+3\lambda-2+2) =$$
$$= -(3+\lambda)(\lambda^2-3\lambda) = -\lambda(3+\lambda)(\lambda-3) = 0 \quad \Rightarrow \lambda_1=0,\ \lambda_2=3,\ \lambda_3=-3$$

Bij eigenwaarde $\lambda_1=0$ hoort dus eigenvector $\underline{x}_1 = \begin{pmatrix}1\\3\\2\end{pmatrix}$. Nu de andere eigenvectoren:

$\lambda_2=3$: $\begin{pmatrix}-1 & 0 & -1\\ 3 & -6 & +3\\ -2 & 0 & -2\end{pmatrix}\underline{x}_2 = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}-1 & 0 & -1\\ 0 & -6 & 0\\ 0 & 0 & 0\end{pmatrix}\underline{x}_2 = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \underline{x}_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$

$\lambda_3=-3$: $\begin{pmatrix}5 & 0 & -1\\ 3 & 0 & 3\\ -2 & 0 & 4\end{pmatrix}\underline{x}_3 = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}5 & 0 & -1\\ 0 & 0 & 19/5\\ 0 & 0 & 0\end{pmatrix}\underline{x}_3 = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \underline{x}_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}$

4 a) $D_2 = x^2-1 \qquad D_3 = x\cdot\begin{vmatrix}x & 1\\ 1 & x\end{vmatrix} + 1\cdot\begin{vmatrix}0 & 1\\ x & 1\end{vmatrix} = x(x^2-1) - x = x^3-2x$

$$D_4 = x\cdot\begin{vmatrix}x & 0 & 1\\ 0 & x & 1\\ 1 & 1 & x\end{vmatrix} - 1\cdot\begin{vmatrix}0 & x & 0\\ 0 & 0 & x\\ 1 & 1 & 1\end{vmatrix} = x\cdot D_3 - 1\cdot 1\cdot\begin{vmatrix}x & 0\\ 0 & x\end{vmatrix} = x^4-2x^2-x^2$$
$$= x^4-3x^2$$

b) $D_n = x^n - (n-1)x^{n-2}$ [Je kunt de relatie $D_n = x\cdot D_{n-1} - x^{n-2}$ vinden, waarna herhaald toepassen van deze regel het antwoord volgt.]

5 a) A en B zijn inverteerbaar, dus $\det A \neq 0$ en $\det B \neq 0$. Omdat $\det(AB) = \det A \cdot \det B \neq 0$, volgt ook dat de matrix AB inverteerbaar is.

b) Er geldt $A\underline{x} = \lambda\underline{x}$ waarbij λ de gegeven eigenwaarde is en $\underline{x}$ de bijbehorende eigenvector. Dus:

$$A^2\underline{x} = A\cdot A\underline{x} = A\cdot\lambda\underline{x} = \lambda\cdot A\underline{x} = \lambda\cdot\lambda\underline{x} = \lambda^2\underline{x}$$

Hieruit volgt het gestelde.