Uitwerking tentamen Wiskunde I — 4 mei 1999.

1. a) $|z^3| = |z|^3 = |-8i| = 8$, dus $|z| = 2$

$\arg z^3 = 3\arg z = \arg(-8i) + 2k\pi = -\frac{\pi}{2} + 2k\pi \quad k \in \mathbb{Z}$

dus $\arg z = -\frac{\pi}{6} + \frac{2}{3}k\pi$, $k \in \mathbb{Z}$. Er zijn 3 oplossingen

$k=0: \quad z_1 = 2e^{-\pi/6\, i} = 2(\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}i) = \sqrt{3} - i$

$k=1: \quad z_2 = 2e^{\frac{1}{2}\pi i} = 2(0 + i) = 2i$

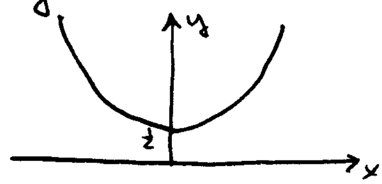
$k=-1: \quad z_3 = 2e^{-\frac{5}{6}\pi i} = 2(-\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}i) = -\sqrt{3} - i$

b) Stel $z = x+iy$. Dan $\mathrm{Im}(z) = y$ en $|z-i| = |x+i(y-1)| = \sqrt{x^2+(y-1)^2}$

Dus $y = \sqrt{x^2+(y-1)^2}$

dus $y^2 = x^2 + (y-1)^2$

dus $y = \frac{1}{2}(1+x^2)$



2. a)
$$\begin{vmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ -z & 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ -x-y-z & 0 & 0 & 0 \end{vmatrix} = -(x+y+z)\begin{vmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} =$$

$$= +(x+y+z)\begin{vmatrix} x & y & z \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = (x+y+z)\begin{vmatrix} x & y & x+y+z \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = (x+y+z)^2 \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = (x+y+z)^2.$$

b)
$$x_3 = \frac{\begin{vmatrix} 0 & 2 & 1 & 4 \\ -2 & 0 & 0 & -1 \\ -3 & -1 & 0 & 1 \\ -4 & 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 3 & 4 \\ -2 & 0 & 1 & -1 \\ -3 & -1 & 0 & 1 \\ -4 & 1 & -1 & 0 \end{vmatrix}} = \frac{\begin{vmatrix} -2 & 0 & -1 \\ -3 & -1 & 1 \\ -4 & 1 & 0 \end{vmatrix}}{(2+3+4)^2} = \frac{1}{81}\left(-2\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -3 & -1 \\ -4 & 1 \end{vmatrix}\right)$$

$$= \frac{1}{81}(2+7) = 1/9.$$

3.
$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 2 & 3 & -4 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1/3 & -2 & -\frac{1}{3} & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -11 & -2 & 6 \\ 0 & 3 & 0 & -6 & 0 & 3 \\ 0 & 0 & 1 & -6 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & -2 & 5 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -6 & -1 & 3 \end{array}\right)$$

dus $B^{-1} = \begin{pmatrix} -9 & -2 & 5 \\ -2 & 0 & 1 \\ -6 & -1 & 3 \end{pmatrix}$

4\. a) $\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x} \underset{=}{\text{L'Hôpital}} \lim_{x\to 0} \frac{1/(2\sqrt{x+3})}{1} = \frac{1}{2\sqrt{3}}$.

b) Eerst zonder grenzen: substitueer $t=e^{-x}$: $\int e^{-x}\sqrt{1-e^{-x}}\,dx \overset{(-dt=e^{-x}dx)}{=}$

$= \int -\sqrt{1-t}\,dt = \frac{2}{3}(1-t)^{\frac{3}{2}} = \frac{2}{3}(1-e^{-x})^{3/2}$. Dus:

$\int_0^\infty e^{-x}\sqrt{1-e^{-x}}\,dx = \lim_{x\to\infty} \frac{2}{3}(1-e^{-x})^{\frac{3}{2}} - \frac{2}{3}(1-e^{0})^{\frac{3}{2}} = \frac{2}{3}$.

5\. a) $\frac{\partial f}{\partial y} = \cos xy - xy\sin xy$; dus $\frac{\partial^2 f}{\partial y^2} = -2x\sin xy - x^2 y\cos xy$

en $\frac{\partial^2 f}{\partial x \partial y} = -2y\sin xy - xy^2\cos xy$. Invullen levert het gevraagde.

b) $\operatorname{div} g(x,y,z) = \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z} = 1+1+1 = 3$

$\operatorname{rot} g(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & y+z & z+x \end{vmatrix} = -i - j - k = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$.

6\. a) Omdat $\int \frac{1}{x}\,dx = \ln x$, vermenigvuldigen we met $e^{\ln x} = x$: $xy' + y = xe^{2x}$

Dus $(xy)' = xe^{2x}$, dus $xy = \int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

dus $y = \frac{1}{2}e^{2x} - \frac{1}{4x}e^{2x} + \frac{C}{x}$. $y(\frac{1}{2}) = 2$; dus $2 = \frac{1}{2}e - \frac{1}{2}e + 2C$

dus $\underline{C=1}$

Totale oplossing: $y(x) = \frac{1}{x}e^{2x}(\frac{1}{2}x - \frac{1}{4}) + \frac{1}{x}$

b) Karakteristieke vgl: $\lambda^2 + 4 = 0$, dus $\lambda = \pm 2i$, dus $y = c_1\cos 2x + c_2 \sin 2x$

c) Particuliere opl: $y_p = Ae^x + B\cos x + C\sin x$, dan $y_p'' = Ae^x - B\cos x - C\sin x$

Invullen: $Ae^x - B\cos x - C\sin x + 4Ae^x + 4B\cos x + 4C\sin x = e^x + \cos x$

dus $A = 1/5$, $B = 1/3$ en $C = 0$. Totale opl: $y(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{5}e^x + \frac{1}{3}\cos x$

7\. $\underline{x}'(t) = A\underline{x}(t)$ met $A = \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$. Eigenwaarden A: $\begin{vmatrix} 4-\lambda & -3 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 5 = 0$

$\lambda_1 = 1$ $\lambda_2 = 5$

Eigenvectoren: $\lambda_1 = 1$ $\begin{pmatrix} 3 & -3 \\ -1 & +1 \end{pmatrix}\underline{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \underline{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda_2 = 5$: $\begin{pmatrix} -1 & -3 \\ -1 & -3 \end{pmatrix}\underline{x}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \underline{x}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Dus $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$