Final Exam: Probability and Statistics (code 400178)

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You are allowed to use a calculator, but not the book or your notes. You can express your numerical answers as fractions.

Each question is worth 2 points (if answered completely and correctly). The total number of points is 40 (corresponding to 20 questions, divided in 8 problems). Give clear answers to as many questions as you can.

Problem 1. A die is thrown twice and the face values that come up are recorded.

a) What is the probability that the two values are different?

b) What is the probability that the sum of the two values is even?

Now consider the events
$A = \{$an odd number comes up at the first throw$\}$,
$B = \{$an odd number comes up at the second throw$\}$,
$C = \{$an odd number comes up exactly once$\}$.

c) List all pairs of events that are independent. Motivate your answer.

d) Are the three events mutually independent? Motivate your answer.

Problem 2. Suppose that $X \sim N(\mu, \sigma^2)$ is a Normal random variables. Let $Y = aX + b$ where $a > 0$.

a) According to what type of distribution is Y distributed?

b) What are the mean and the variance of Y?

Problem 3. A die is thrown N times and the face values that come up are recorded. Suppose that k different values are recorded. (For example, if $N = 3$ and the outcomes are 2, 6, 6, then $k = 2$.)

a) For a given N, what is the probability that $k = 1$?

b) If $k = 1$, what is the maximum likelihood estimate for N? Motivate your answer.

Problem 4. Consider two systems, the first made of a single unit which fails with probability p_1, the second made of two independent units, each of which fails with probability p_2. The second system fails only if both its units fail.

a) How small should p_1 be compared to p_2 to make the first system more reliable that the second one?

Consider now a system with n independent units, each of which fails with probability p. The system fails only if k or more units fail.

b) What is the probability that the system fails? (You can express your answer as a sum.)

Problem 5. A box contains a white ball and two red balls. A ball is drawn, and then the same ball and another ball of the same color are placed back in the box. Finally, a second ball is drawn.

a) What is the probability that the second ball drawn is white?

b) If the second ball drawn is white, what is the probability that the first ball drawn was red?

Problem 6. A radioactive source contains $N = 3 \times 10^{23}$ atoms. Each atom has a probability $p = 10^{-23}$ to emit an alpha particle in the next minute. Assume that the atoms emit alpha particles independently of each other and that each atom can emit at most one alpha particle in the next minute.

a) What is the expected number of alpha particles emitted by the radioactive source in the next minute?

b) Given the assumptions (i.e., the large number N of atoms and the small probability p that an atom emits an alpha particle in the next minute) what distribution can you use to compute (approximately) the probability π_n that n alpha particles are emitted by the radioactive source in the next minute? Use it to compute π_0, π_1 and π_2. (If you don't want to use a calculator, here and in the next question you can replace the number $1/e$ with $1/3$.)

c) What is the probability that 3 or more alpha particles are emitted in the next minute?

d) Suppose that no alpha particle has been emitted in the last forty seconds. What is the probability that no alpha particle is emitted in the next twenty seconds?

Problem 7. Let X be a uniform random variable on $[0, 1]$.

a) Find $E(X^n)$ for $n = 1, 2, 3 \dots$.

b) Are $E(X^n)$ and $E(X)^n$ the same for any $n > 1$? When they are not, which is greater?

c) Find the variance of X^n for $n = 1, 2, 3 \dots$.

Problem 8. A company has manufactured certain objects and has printed a serial number on each manufactured object. The serial numbers start at 1 and end at N, where N is the number of objects that have been manufactured. One of these objects is selected at random, and the serial number of that object is 999.

a) What is the maximum likelihood estimate of N?