Final Exam: Probability and Statistics (code 400178)

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You are allowed to use a calculator, but the calculations involved in solving the problems can all be carried out without one. You can express your numerical answers as fractions. In carrying out the calculations of Problems 4 and 5, you can for simplicity replace the number $1/e$ with $1/3$.

Each question is worth 2 points (if answered completely and correctly). The total number of points is 40 (corresponding to 20 questions, divided in 6 problems). Give clear answers to as many questions as you can.

Problem 1. In conducting an experiment, a temperature is measured several times and it is found that the values are distributed according to a Normal distribution with $\mu = 15$ Celsius and $\sigma = 0.1$ Celsius. To convert a Celsius temperature to Fahrenheit, you have to multiply by $9/5$ and then add 32.

a) What is the distribution of the temperature expressed in Fahrenheit?

b) What are the expected value and the variance of the temperature expressed in Fahrenheit?

Problem 2. Let X be a uniform random variable on $[a, b]$.

a) What is the density function $f(x)$ of X?

b) Calculate the expected value and the variance of X.

c) Find x_0 such that $P(X > x_0) = 1/3$.

d) A line segment of length 1 is cut once at random. What is the probability that the longer piece is more than twice the length of the shorter piece?

Problem 3. Consider a dice with the faces 2, 3, 5 colored red and the other faces colored green.

a) What is the expected number of times you have to roll the dice to get a red face with an even number on it?

b) If the dice is rolled n times, what is the probability of obtaining at least m times a red face with an even number on it? ($m \leq n$)

c) The dice is rolled twice. Are the events $A = \{\text{two even numbers come out}\}$ and $B = \{\text{red comes out at least once}\}$ independent? Compute $P(A|B)$.

d) The dice is rolled three times. What is the probability of getting red two or more times given that red came out at least once?

e) The dice is rolled three times. What is the probability of getting red two or more times given that three even numbers came out?

f) Each of 3 persons rolls the dice once. What is the probability that two or more persons obtain the same number?

Problem 4. Assume that the waiting time of a job in a printer's queue can be modelled by an exponential random variable with density

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

where time is measured in minutes.

a) What is the probability that a job has to wait for longer than a time $\tau = 1/\lambda$?

b) Suppose that you send a job to the printer at 10:30. At 10:45 your job has not been printed yet. How much longer do you have to wait (approximately) to have a probability of approximately two thirds that your job has been printed? Express your answer in terms of λ.

c) At 11:00 your job has not been printed yet. Answer question b) again. Express your answer in terms of λ.

Problem 5. A piece of radioactive material contains $N = 2 \times 10^{20}$ nuclei. Each nucleus has a probability $p = 10^{-20}$ to emit an alpha particle in the next minute. Assume that the nuclei emit alpha particles independently of each other and that each nucleus can emit at most one alpha particle in the next minute.

a) What is the expected number of alpha particles emitted by the piece of radioactive material in the next minute?

b) Given the assumptions, the large number N of nuclei and the small probability p that a nucleus emits an alpha particle in the next minute, what distribution can you use to compute (approximately) the probability π_n that n alpha particles are emitted by the piece of radioactive material in the next minute? Use it to compute π_0, π_1 and π_2.

c) What is the probability that 3 or more alpha particles are emitted in the next minute?

Problem 6. Consider a Poisson process on the line with parameter λ. The interval $[0, 1]$ is partitioned into 10 subintervals $I_1, I_2, \dots, I_{10}$ of equal length (1/10 of the length of the unit interval $[0, 1]$).

a) What is the probability that, moving from 0 to 1, the first point of the Poisson process is encountered in the subinterval I_k?

b) Each subinterval I_k is assigned a number X_k equal to k times the number N_k of points of the Poisson process in I_k. Calculate the expected value of the random variable $X = \sum_{k=1}^{10} X_k$.