HERTENTAMEN

# Kansrekening en Statistiek – 4I

WEDNESDAY 31 AUGUST 2005

INSTRUCTIONS

- Write your name and studentnumber on *every* page
- Motivate all answers clearly
- Use of a calculator will *not* be necessary and is *not* allowed
- Use of Rice's book is allowed
- If you finish the exam early, hand in your work and leave *quietly*
- Do *not* attempt to cheat or confer with your fellow students

Points per question *(Grade = total/8 + 1)*

| | | | | | |
|---|---|---|---|---|---|
| 1(a): | 5 | 2(a): | 5 | 3(a): | 5 |
| 1(b): | 5 | 2(b): | 5 | 3(b): | 5 |
| 1(c): | 5 | 2(c): | 6 | 3(c): | 5 |
| 1(d): | 6 | 2(d): | 5 | 3(d): | 5 |
| | | | | 3(e): | 5 |
| | | | | 3(f): | 5 |

Good luck!

[DO NOT TURN THIS PAGE UNTIL SO INSTRUCTED]

## PROBLEM 1

Let $X$ in $\mathbb{R}$ be continuously distributed with probability density function $f_X : \mathbb{R} \to \mathbb{R}$ of the form:

$$f_X(x) = \begin{cases} Cx^2(2-x)^2, & \text{if } 0 \leq x \leq 2, \\ 0, & \text{if } x < 0 \text{ or } x > 2. \end{cases}$$

a. *(5 points)*
Calculate the normalization constant $C$.

b. *(5 points)*
Show that $E(X) = 1$.

c. *(5 points)*
Show that the second moment $E(X^2)$ satisfies

$$E(X^2) = \frac{128}{105}C,$$

and, using your answer under part *a.*, show that $\mathrm{Var}(X) = 8/7$.

d. *(6 points)*
Give the distribution function $F_X : \mathbb{R} \to [0,1]$ of $X$.
Based on your answer, demonstrate that the median of $X$ equals 1.

## PROBLEM 2

Let $X, Y$ be two continuous random variables, jointly distributed according to a joint probability density function $f_{XY} : \mathbb{R}^2 \to \mathbb{R}$,

$$f_{XY}(x,y) = \begin{cases} Ce^{-\lambda(x+y)}, & \text{if } x \geq 0 \text{ and } y \geq 0, \\ Ce^{-\lambda(x-y)}, & \text{if } x \geq 0 \text{ and } y < 0, \\ Ce^{-\lambda(-x+y)}, & \text{if } x < 0 \text{ and } y \geq 0, \\ Ce^{\lambda(x+y)}, & \text{if } x < 0 \text{ and } y < 0. \end{cases}$$

depending on a parameter $\lambda > 0$.

a. *(5 points)*
Calculate the normalization constant $C$
*(NB: your answer will be an expression involving $\lambda$).*

b. *(5 points)*

Show that the marginal probability density function $f_X : \mathbb{R} \to \mathbb{R}$ for $X$ has the following form:

$$f_X(x) = \begin{cases} \frac{\lambda}{2}\, e^{-\lambda x}, & \text{if } x \geq 0, \\ \frac{\lambda}{2}\, e^{\lambda x}, & \text{if } x < 0. \end{cases}$$

c. *(6 points)*

Show that the random variables $X$ and $Y$ are independent (motivating your answer by an equality of probability density functions).

What relation does the independence of $X$ and $Y$ provide between $\mathrm{Var}(X+Y)$, $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$?

d. *(5 points)*

Show that:

$$\mathrm{Var}(X+Y) = \frac{4}{\lambda^2}.$$

*(NB: You may use that $\int_0^\infty u^n e^{-u}\, du = n!$.)*

PROBLEM 3

Consider an experiment in which we measure $X$, a random variable with a distribution depending on an unknown parameter $\kappa > 0$. The probability density function $f_\kappa : \mathbb{R} \to \mathbb{R}$ for $X$ is given by:

$$f_\kappa(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{\kappa\sqrt{\pi}}\, x^{-1/2}\, e^{-\frac{x}{\kappa^2}}, & \text{if } x \geq 0. \end{cases}$$

Assume that we have an *i.i.d.* sample $X_1, X_2, \ldots, X_n$ of repeated measurements of $X$. Parts *a.* and *b.* of this problem concern the moment estimator for $\kappa$.

a. *(5 points)*

By partial integration, show that:

$$\int_0^\infty x^{1/2}\, e^{-x/\kappa^2}\, dx = \frac{\kappa^2}{2} \int_0^\infty x^{-1/2}\, e^{-x/\kappa^2}\, dx.$$

Use this equality to show that $E(X) = \kappa^2/2$.

b. *(5 points)*

Give the first moment equation and derive the form of the moment estimator $\tilde{\kappa}$ for the parameter $\kappa$.

Indicate clearly in which step of your derivation the restriction $\kappa > 0$ plays a role.

Next, we turn to the maximum-likelihood estimator of $\kappa$. As it turns out, it is conventient to use, instead of $\kappa$ itself, its square $\theta = \kappa^2$ as a parameter. This does not change the probability density functions in the model, only the parametrization. To obtain the maximum-likelihood estimator for $\kappa$, we shall solve the maximum-likelihood problem for $\theta$ first and then derive the maximum-likelihood estimator for $\kappa$.

c. *(5 points)*
What is the domain of the new parameter $\theta$?
Give the form for the probability density function $f_\theta : \mathbb{R} \to \mathbb{R}$ parametrized by the parameter $\theta$.
Also give the log-likelihood $\ell(\theta)$ based on the *i.i.d.* sample $X_1, \ldots, X_n$.

d. *(5 points)*
Write down the first derivative $\dot{\ell}(\theta)$ of $\ell(\theta)$ with respect to $\theta$.
Solve the equation $\dot{\ell}(\theta) = 0$ to show that the maximum-likelihood estimator $\hat{\theta}$ for $\theta$ equals:

$$\hat{\theta}(X_1, \ldots, X_n) = 2\bar{X}.$$

e. *(5 points)*
Based on part *d.*, give the maximum-likelihood estimator $\hat{\kappa}$ for $\kappa$.
Explain your answer clearly.
*(Hint: this does not require any calculation: extrema of a function are parametrization-invariant.)*

f. *(5 points)*
Is the maximum-likelihood estimator $\hat{\theta}$ for $\theta$ biased or unbiased?
Can we conclude from this that the maximum-likelihood estimator $\hat{\kappa}$ for $\kappa$ is baised/unbiased?