TENTAMEN

Kansrekening en Statistiek – 4I

WEDNESDAY 1 JUNE 2005

INSTRUCTIONS

- Write your name and studentnumber on *every* page
- Motivate all answers clearly
- Use of a calculator will *not* be necessary and is *not* allowed
- Use of Rice's book is allowed
- If you finish the exam early, hand in your work and leave *quietly*
- Do *not* attempt to cheat or confer with your fellow students

Points per question *(Grade = total/10 + 1)*

1(a):	5	2(a):	5	3(a):	5	4(a):	5
1(b):	5	2(b):	5	3(b):	5	4(b):	5
1(c):	5	2(c):	5	3(c):	5	4(c):	5
1(d):	5			3(d):	5	4(d):	5
1(e):	5					4(e):	5
1(f):	5						

Good luck!

[DO NOT TURN THIS PAGE UNTIL SO INSTRUCTED]

PROBLEM 1

In this problem we consider two random experiments conducted with a (standard) deck of 52 cards. In the first experiment, we shuffle the deck thoroughly, draw a card (noting the outcome), replace the card in the deck, shuffle thoroughly again and draw a second card from the deck.

a. Describe the samplespace Ω_1 for the first experiment.
   Argue that Ω_1 consists of 1144 points.

b. Are the first and second draws independent in the first experiment?
   Motivate your answer clearly.

c. What is the probability that the suits of the first and second cards are the same in the first experiment?

In the second experiment, we shuffle the deck thoroughly, draw a card, we do *not* replace it in the deck, shuffle the remainder of the deck thoroughly and we draw a second card from the deck.

d. Describe the samplespace Ω_2 for the second experiment.
   How does Ω_2 differ from Ω_1?
   Argue that Ω_2 consists of 1092 points.

e. Are the first and second draws independent in the second experiment?
   Motivate your answer clearly.

f. What is the probability that the suits of the first and second cards are the same in the second experiment?
   Explain in words why this probability is smaller than the answer under *c.*.

PROBLEM 2

Let X in $\mathbb{R}$ be continuously distributed with probability density function $f_X : \mathbb{R} \to \mathbb{R}$ of the form:

$$f_X(x) = \begin{cases} Cx(3-x), & \text{if } 0 \leq x \leq 3, \\ 0, & \text{if } x < 0 \text{ or } x > 3. \end{cases}$$

a. Show that the normalization constant C equals $2/9$.

b. Calculate the expectation $E(X)$, second moment $E(X^2)$ and $\mathrm{Var}(X)$.

c. Give the distribution function $F_X : \mathbb{R} \to [0, 1]$ of X.
Based on your answer, demonstrate that the median of X equals $3/2$.

PROBLEM 3

Let X, Y be two continuous random variables, jointly distributed according to the probability density function $f_{XY} : \mathbb{R}^2 \to \mathbb{R}$, given by:

$$f_{XY}(x, y) = \begin{cases} Ce^{-\lambda(x+y)}, & \text{if } x \geq 0 \text{ and } y \geq 0, \\ 0, & \text{if } x < 0 \text{ or } y < 0. \end{cases}$$

depending on a parameter $\lambda > 0$.

a. Calculate the normalization constant C
(Note: your answer will be an expression involving λ).

b. Show that the marginal probability density function $f_X : \mathbb{R} \to \mathbb{R}$ for X has the following form:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

c. Are the random variables X and Y independent?
Motivate your answer by an equality.

d. Calculate the expectation $E(X + Y)$.

PROBLEM 4

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed continuous random variables. Marginally, the X_i are all distributed like $X \sim P_\theta$ with probability density function $f_X : \mathbb{R} \to \mathbb{R}$ of the form:

$$f_\theta(x) = \begin{cases} \frac{1}{\theta}\, e^{-\frac{x}{\theta}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Here, $\theta > 0$ is an unknown constant. Throughout this problem, we use the identify:

$$\int_0^\infty u\, e^{-u}\, du = 1. \tag{1}$$

a. Using a substitution of variables and identity (1), show that:

$$E(X) = \theta.$$

b. Use the method of moments to derive an estimator $\tilde{\theta}$ for θ based on the sample $X_1, X_2, \ldots, X_n$.

c. Derive the maximum-likelihood estimator $\hat{\theta}$ for θ based on the sample X_1, X_2, $\ldots$, X_n.
Clearly state what you are doing at every step.

Taking the Bayesian perspective on this estimation problem, we view θ as a continuous random variable taking values in $(0, \infty)$, with (prior) density:

$$g(\theta) == \begin{cases} \frac{1}{\theta^2} e^{-\frac{1}{\theta}}, & \text{if } \theta > 0, \\ 0, & \text{if } \theta \leq 0. \end{cases}$$

d. Write down the expression for the posterior density based on the sample $X_1, X_2, \ldots, X_n$. *(Do not do any extensive calculations)*
What is the most popular procedure to extract an estimator for θ from the posterior?

e. Show that the posterior mean equals $\bar{X}_n + 1/n$. *(Hint: Substitute $\lambda = 1/\theta$, rescale by a factor $1 + \sum_i X_i$, then use the identity $\int_0^\infty u^n e^{-u}\, du = n!$.)*