Final exam
Introduction to Differential Manifolds
for students Mathematics and Physics

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Instructions: 5 problems; *motivate all answers.*

Score: 1 (20 pts), 2 (20 pts), 3 (20 pts), 4 (20 pts), 5 (20 pts).

1. Given the 2-torus

$$\mathbf{T}^2 = \{(x,y,z,w) \; : \; x^2+y^2=1, \quad z^2+w^2=1\} \subset \mathbf{R}^4.$$

   (a) Consider coordinates (U,φ), with $U = \{p \in \mathbf{T}^2 \; : \; x,y>0, \quad z,w>0\}$, and $(\theta,\eta) = \varphi(p) = (\tan^{-1}(y/x), \tan^{-1}(w/z))$. Compute $T_p\mathbf{T}^2$ for $p \in U$.

   (b) Given the map $f : \mathbf{T}^2 \to \mathbf{R}^2$ defined by

$$f(x,y,z,w) = (xy, zw).$$

   Compute the pushforward $f_* : T_p\mathbf{T}^2 \to \mathbf{R}^2$ for $p \in U$.

   (c) Find the points where f_* is not of maximal rank.

2. Given the set

$$M = \{(x,y) \; : \; x^6+y^6=1\} \subset \mathbf{R}^2.$$

   (a) Construct a differentiable atlas (smooth transition mappings) to show that M is a smooth manifold.

   (b) Prove that M is diffeomorphic to the circle $S^1 = \{(x,y) \; : \; x^2+y^2=1\} \subset \mathbf{R}^2$ with respect to an appropriately chosen differentiable atlas for S^1.

3. Show that S^2 is orientable and give an orientation.

4. Consider the 2-torus $\mathbf{T}^2$ in Problem 1 and the 1-form

$$\sigma = (x^3-3x)dy - y^3dx + z^3dw - (w^3-3w)dz$$

   on $\mathbf{T}^2$.

   (a) Show that σ is a smooth 1-form on $\mathbf{T}^2$.

   (b) Verify that σ is closed.

   (c) Consider the mapping $g : S^1 \to \mathbf{T}^2$ given by

$$g(p,q) = (p, q, (q-p)/\sqrt{2}, (p+q)/\sqrt{2}),$$

   and compute the pullback form $g^*\sigma$ on S^1.

(d) Is σ an exact 1-form?

5. Given the cylinder

$$M = \{(x, y, z) \ : \ x^2 + y^2 = 1, \quad 0 \leq z \leq 1\},$$

with the standard orientation $\mathcal{O}$.

(a) Determine the induced, or Stokes orientation $\partial\mathcal{O}$ on the boundary ∂M.

(b) Given the 2-form

$$\sigma = dx \wedge dy + dy \wedge dz + dx \wedge dz,$$

on M, show that σ is exact.

(c) Use Stokes' Theorem to compute the integral

$$\int_M \sigma.$$

Good luck