Faculteit der Exacte Wetenschappen	Introduction Partial Differential Equations
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Grading: Each item in 1 and 2 one point. Exercises 3, 4 and 5 one point each. Total 12 (2 bonus points). Your grade: a weighted average of your total score and your overall home work score, truncated at 10. Maximal weight homework: $\frac{1}{5}$.

1. This exercise concerns various techniques for the problem

$$-u''(x) = f(x) \quad (0 \leq x \leq \pi), \quad u(0) = u(\pi) = 0. \tag{1}$$

a) Derive a solution formula of the form

$$u(x) = \int_0^\pi G(x,y)f(y)dy$$

for the problem in (1). This requires you to determine explicitly the Green's function $G(x,y)$.

b) Derive the Fourier sinus series expansion for

$$f(x) = 1 \sim \sum_{n=1}^\infty a_n \sin nx, \qquad x \in [0,\pi].$$

and apply Parceval's identity for $\int_0^\pi f(x)^2 dx$ to derive an expression for

$$\sum_{n=1}^\infty a_n^2.$$

c) Let u be the solution of the problem in (1) with $f(x) = 1$. Derive the Fourier sinus series expansion of u, and evaluate the sum in $x = \frac{\pi}{2}$.

d) (bonus) Assuming u and f smooth, multiply the equation in (1) by a smooth function v with $v(0) = v(\pi) = 0$, integrate and derive that there are two inner products $((\cdot,\cdot))$ and $(\cdot,\cdot)$ such that

$$((u,v)) = (f,v).$$

e) (bonus) We say that ϕ is an eigenfunction with real eigenvalue λ if $u = \phi$ is a solution of the problem in (1) with $f = \lambda\phi$. Show that different real eigenvalues λ and μ have eigenfunctions ϕ and ψ with

$$\int \phi(x)\psi(x)dx = 0.$$

2. Let $\mathrm{N} > 1$. For a smooth function $\Psi : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ of the form $\Psi(x_1, \dots, x_{\mathrm{N}}) = R(r)$, with $r^2 = x_1^2 + \dots + x_{\mathrm{N}}^2$, the Laplacian is given by

$$\Delta\Psi = \frac{d^2R}{dr^2} + \frac{\mathrm{N}-1}{r}\frac{dR}{dr} = \frac{1}{r^{\mathrm{N}-1}}\frac{d}{dr}(r^{\mathrm{N}-1}\frac{dR}{dr}) \tag{2}$$

If $\Psi(_1, \dots, x_{\mathrm{N}}) = R(r)$ is a smooth solution of $\Delta\Psi + \Psi = 0$, then

$$R(r) = \sum_{n=0}^{\infty} a_n r^n.$$

a) For $\mathrm{N} = 2$ you have seen that $(n+2)^2 a_{n+2} + a_n = 0$. Derive the recurrence relation for general $\mathrm{N} > 1$.

b) Explain why $a_1 = 0$ and why there is only one smooth solution of

$$\frac{1}{r^{\mathrm{N}-1}}\frac{d}{dr}\left(r^{\mathrm{N}-1}\frac{dR}{dr}\right) + R = 0. \tag{3}$$

with $R(0) = 1$. Denote this solution by $J(r)$.

c) The solution $J(r)$ above is defined for all r. The dependence on N is suppressed in the notation. This nice function is oscillating and has a countable sequence of zero's $r_1 < r_2 < r_3 < \dots \to \infty$. What is this solution in the special case that $N = 1$?

3. Let $B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ be the unit ball in $\mathbb{R}^3$. In this exercise we consider the damped wave equation

$$u_{tt} + bu_t = \Delta u \tag{4}$$

for functions $u(x, y, z, t)$ defined on $B \times \mathbb{R}$ with boundary condition $u = 0$ on $\partial B \times \mathbb{R}$. Here b is a positive parameter. Separate the time variable from the spatial variables by putting $u(x, y, z, t) = T(t)\Psi(x, y, z)$ and derive that Ψ must be a solution of the Helmholtz equation

$$\Delta\Psi + k\Psi = 0 \text{ on } B \quad \text{with} \quad \Psi = 0 \text{ on } \partial B, \tag{5}$$

where k is still to be determined. Give the corresponding differential equation for $T(t)$. What is the condition on b and k needed to have oscillating solutions $T(t)$?

4. Let $\Omega \subset \mathbb{R}^{\mathrm{N}}$ be open and connected. It was proved in the course that a continuous function $u : \Omega \to \mathbb{R}$ which satisfies

$$u(x_0) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(x)dx$$

for all closed balls $B_r(x_0)$ (with center x_0 and radius r) in Ω, is in fact smooth and satisfies $\Delta u = 0$ in Ω. Here $|B_r(x_0)| = \int_{B_r(x_0)} 1dx$ is the N-dimensional measure of $B_r(x_0)$. Show that if such a harmonic function u is nonnegative on Ω, and $u(x_0) = 0$ for some $x_0 \in \Omega$, then $u(x) = 0$ for all $x \in \Omega$.

5. Use polar coordinates ($x = r\cos\phi, y = r\sin\phi$) to solve $-\Delta u = 1$ on the unit disk, with $u(x, y) = xy$ on boundary circle $x^2 + y^2 = 1$.