Make-up exam	Afdeling Wiskunde
Introduction Partial Differential Equations	Faculteit der Exacte Wetenschappen
for students Mathematics and Physics	Vrije Universiteit Amsterdam

Date: Tuesday February 6, 2007, 18:30– 21:15 (3 hours)
Instructions: 6 problems; *motivate all answers.*
No calculators, no books, no formula sheets.

Scores: 1 (30 pts), 2 (35 pts), 3 (35 pts), 4 (35 pts), 5 (25 pts), 6 (40 pts)

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1. (a) Calculate the Fourier series of $|x|$ on $[-\pi, \pi]$.
   (b) Compute the sine Fourier series of $|x|$ on $[0, \pi]$.
   (c) Explain the different rates of convergence of the Fourier series in (a) and (b).

2. Consider the fourth order equation

$$u_t = -u_{xxxx}, \quad x \in [0, \pi], \quad t > 0$$

with initial condition $u(0, x) = f(x) \in C^0([0, \pi])$ and boundary conditions $u(t, 0) = u_{xx}(t, 0) = 0$, and $u(t, \pi) = u_{xx}(t, \pi) = 0$.

   (a) Give the general solution using separation of variables.
   (b) Derive the formulas for the Fourier coefficients in terms of the initial function f.
   (c) Let $f(x) = x$ on $[0, \pi]$. Compute the solution $u(t, x)$.
   (d) Prove that $\lim_{t\to\infty} u(t, x) = 0$ uniformly in $x \in [0, \pi]$.

3. Consider the eigenvalue problem

$$L\varphi = \varphi_{x_1 x_1} + \varphi_{x_2 x_2} + 25\varphi = \lambda\varphi, \qquad (x_1, x_2) \in D = (0, \pi) \times (0, \pi),$$

with boundary conditions $u|_{\partial D} = 0$.

   (a) Show, using separation of variables, that the eigenvalues and eigenfunctions are

$$\lambda_{n,m} = -(n^2 + m^2 - 25), \quad \varphi_{n,m} = \frac{2}{\pi} \sin(nx_1) \sin(mx_2), \quad n, m \in \{1, 2, \cdots\}.$$

   (b) Let $FS(f) = \frac{2}{\pi} \sum_{n,m} c_{n,m} \sin(nx_1) \sin(mx_2)$ be the Fourier expansion of a given function $f(x_1, x_2)$.
   Show that

$$c_{n,m} = \frac{2}{\pi} \int_0^\pi \int_0^\pi f(x_1, x_2) \sin(nx_1) \sin(mx_2) dx_1 dx_2.$$

   (c) Compute the null-space of L, i.e. all functions φ such that $L\varphi = 0$. What is the dimension of this space?

4. Consider the function

$$\delta_\epsilon(x) = \frac{1}{2\epsilon}, \quad x \in (-\epsilon, \epsilon),$$

and $\delta_\epsilon(x) = 0$ for $x \notin (-\epsilon, \epsilon)$.

(a) Given $f \in C_0^\infty(\mathbf{R})$, with $|f^{(k)}(x)| \leq 1$ for all $x \in \mathbf{R}$, and all $k \in \mathbf{N}$. Expand the integral

$$\int_{\mathbf{R}} \delta_\epsilon(x) f(x) dx$$

in ϵ (Hint: use the Taylor expansion for f around $x = 0$).

(b) Compute the limit $\epsilon \to 0$. The limit will be denoted $\delta(x)$.

(c) Use answers in (a) and (b), to compute the integral

$$\int_{\mathbf{R}} \sin(x)\delta(x - \frac{\pi}{2})dx.$$

5. (a) Calculate the general solution of

$$u_x + 2u_y + 4u = 0$$

(Hint: don't use separation of variables and use an appropriate substitution!).

(b) Find the unique solution that satisfies the initial condition $u(0, y) = 1$.

6. Consider the ordinary differential equation

$$u'' = f, \quad t \in (0, 1),$$

with $u(0) = u'(1) = 0$, and f continuous on $[0, 1]$.

(a) Formulate the differential equation for the Green's function $G(t; s)$ (Hint: use the δ-function).

(b) Compute the Green's function $G(t; s)$ in the representation formula

$$u(t) = \int_0^1 G(t; s)f(s)ds$$

(Hint: use the equation found in (a), or use the variation of parameters method).

(c) Show that $G(t; s)$ is continuous and symmetric with respect to t and s, i.e. $G(t; s) = G(s; t)$.

(d) Find the solution u when the boundary conditions are $u(0) = a$, $u'(1) = b$ (Hint: use Green's identity in order to include the boundary conditions).

The cosine Fourier series for f are $\sum_{n=0}^{\infty} a_n \cos(nx)$, $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)dx$, $n \geq 1$, and $a_0 = \frac{1}{\pi}\int_o^\pi f(x)dx$.

The sine Fourier series for f are $\sum_{n=1}^{\infty} a_n \sin(nx)$, $a_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx$, $n \geq 1$.

The Fourier series for f are $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$, and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)$.

Good luck