Midterm/Final exam Afdeling Wiskunde
Introduction Partial Differential Equations Faculteit der Exacte Wetenschappen
for students Mathematics and Physics Vrije Universiteit Amsterdam

Date: Thursday December 21, 2006, 8:45– 10:30/11:30 (2/3 hours)
Instructions: 3/5 problems; *motivate all answers.*
No calculators, no books, no formula sheets.
For midterm exam 2: problems 1, 3 and 4, total time 2 hours.
For the final exam: problems 1 through 5, total time 3 hours.

Midterm 2: 1 (35 pts), 3 (30 pts), 4 (35 pts).
Final: 1 (35 pts), 2 (35 pts), 3 (30 pts), 4 (35 pts), 5 (15 pts).

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1. Consider the eigenvalue problem

$$L\varphi = \varphi_{x_1x_1} + \varphi_{x_2x_2} + 25\varphi = \lambda\varphi, \qquad (x_1, x_2) \in D = (0,\pi) \times (0,\pi),$$

with boundary conditions $u|_{\partial D} = 0$.

(a) Show, using separation of variables, that the eigenvalues and eigenfunctions are

$$\lambda_{n,m} = -(n^2 + m^2 - 25), \quad \varphi_{n,m} = \frac{2}{\pi}\sin(nx_1)\sin(mx_2), \quad n, m \in \{1, 2, \cdots\}.$$

(b) Let $FS(f) = \frac{2}{\pi}\sum_{n,m} c_{n,m}\sin(nx_1)\sin(mx_2)$ be the Fourier expansion of a given function $f(x_1, x_2)$.
Show that

$$c_{n,m} = \frac{2}{\pi}\int_0^\pi\int_0^\pi f(x_1, x_2)\sin(nx_1)\sin(mx_2)dx_1dx_2.$$

(c) Compute the null-space of L, i.e. all functions φ such that $L\varphi = 0$. What is the dimension of this space?

2. Consider the second order equation

$$u_t = u_{xx} + \frac{1}{4}u, \quad x \in [0, \pi], \quad t > 0$$

with initial condition $u(0, x) = f(x) \in C^0([0, \pi])$ and boundary conditions $u(t, 0) = 0$, and $u_x(t, \pi) = 0$.

(a) Give the general solution using separation of variables.

(b) Derive the formulas for the Fourier coefficients in terms of the initial function f.

(c) Let $f(x) = 1$ on $[0, \pi]$. Compute the solution $u(t, x)$.

(d) Prove that $\lim_{t\to\infty} u(t, x) = \frac{4}{\pi}\sin(x/2)$ uniformly in $x \in [0, \pi]$.

3. Consider the function

$$\delta_\epsilon(x) = \frac{1}{2\epsilon}, \quad x \in (-\epsilon, \epsilon),$$

and $\delta_\epsilon(x) = 0$ for $x \notin (-\epsilon, \epsilon)$.

(a) Given $f \in C_0^\infty(\mathbf{R})$, with $|f^{(k)}(x)| \leq 1$ for all $x \in \mathbf{R}$, and all $k \in \mathbf{N}$. Expand the integral

$$\int_{\mathbf{R}} \delta_\epsilon(x) f(x) dx$$

in ϵ (Hint: use the Taylor expansion for f around $x = 0$).

(b) Compute the limit $\epsilon \to 0$. The limit will be denoted $\delta(x)$.

(c) Use answers in (a) and (b), to compute the integral

$$\int_{\mathbf{R}} \sin(x)\delta(x - \frac{\pi}{2})dx.$$

4. Consider the ordinary differential equation

$$u'' = f, \quad t \in (0, 1),$$

with $u(0) = u'(1) = 0$, and f continuous on $[0, 1]$.

(a) Formulate the differential equation for the Green's function $G(t; s)$ (Hint: use the δ-function).

(b) Compute the Green's function $G(t; s)$ in the representation formula

$$u(t) = \int_0^1 G(t; s) f(s) ds$$

(Hint: use the equation found in (a), or use the variation of parameters method).

(c) Show that $G(t; s)$ is continuous and symmetric with respect to t and s, i.e. $G(t; s) = G(s; t)$.

(d) Find the solution u when the boundary conditions are $u(0) = a$, $u'(1) = b$ (Hint: use Green's identity in order to include the boundary conditions).

5. (a) Calculate the general solution of

$$u_x + u_y + u = 0$$

(Hint: substitute $v = e^{\frac{1}{2}(x+y)}u$ and don't use separation of variables!).

(b) Find the unique solution that satisfies the initial condition $u(0, y) = e^y$.

The cosine Fourier series for f are $\sum_{n=0}^{\infty} a_n \cos(nx)$, $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)dx$, $n \geq 1$, and $a_0 = \frac{1}{\pi}\int_0^\pi f(x)dx$.

The sine Fourier series for f are $\sum_{n=1}^{\infty} a_n \sin(nx)$, $a_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx$, $n \geq 1$.

The Fourier series for f are $\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$, and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)$.

Good luck