Midterm/Final exam
Introduction Partial Differential Equations
for students Mathematics and Physics

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Date: Thursday December 22, 2005, 9:30– 11:30/12:30 (2/3 hours)
Instructions: 4/6 problems; *motivate all answers.*
No calculators, no books, no formula sheets.
For midterm exam 2: problems 1 through 4, total time 2 hours.
For the final exam: problems 1 through 6, total time 3 hours.

Midterm 2: 1 (30 pts), 2 (30 pts), 3 (20 pts), 4 (20 pts).
Final: 1 (20 pts), 2 (20 pts), 3 (15 pts), 4 (15 pts), 5 (15 pts), 6 (15 pts).

1. Consider the second order equation

$$u_t = u_{xx} + \frac{1}{4}u, \quad x \in [0,\pi], \quad t > 0$$

with initial condition $u(0,x) = f(x) \in C^0([0,\pi])$ and boundary conditions $u(t,0) = 0$, and $u_x(t,\pi) = 0$.

   (a) Give the general solution using separation of variables.
   (b) Derive the formulas for the Fourier coefficients in terms of the initial function f.
   (c) Let $f(x) = 1$ on $[0,\pi]$. Compute the solution $u(t,x)$.
   (d) Prove that $\lim_{t\to\infty} u(t,x) = \frac{4}{\pi}\sin(x/2)$ uniformly in $x \in [0,\pi]$.

2. Given de elliptic equation

$$u_{x_1x_1} + u_{x_2x_2} - u = f(x_1,x_2), \qquad (x_1,x_2) \in D = (0,\pi)\times(0,\pi),$$

with boundary conditions $u|_{\partial D} = 0$.

   (a) Formulate the equation for the Green's function.
   (b) Consider the eigenvalue problem

$$\varphi_{x_1x_1} + \varphi_{x_2x_2} - \varphi = \lambda\varphi, \qquad (x_1,x_2) \in D = (0,\pi)\times(0,\pi),$$

   with the above boundary conditions. Show, using separation of variables, that the eigenvalues and eigenfunctions are

$$\lambda_{n,m} = -(n^2 + m^2 + 1), \quad \varphi_{n,m} = \frac{2}{\pi}\sin(nx_1)\sin(mx_2), \quad n,m \in \{1,2,\cdots\}.$$

   (c) Let $FS(f) = \frac{2}{\pi}\sum_{n,m} c_{n,m}\sin(nx_1)\sin(mx_2)$ be the Fourier expansion of a given function $f(x_1,x_2)$.
   Show that

$$c_{n,m} = \frac{2}{\pi}\int_0^\pi\int_0^\pi \sin(nx_1)\sin(mx_2)dx_1dx_2.$$

(d) Use the eigenfunction expansion to compute $G(x,y)$, and verify that G is symmetric.

3. Consider the function

$$\delta_\epsilon(x) = \frac{1}{2\epsilon}, \quad x \in (-\epsilon, \epsilon),$$

and $\delta_\epsilon(x) = 0$ for $x \notin (-\epsilon, \epsilon)$.

(a) Given $f \in C_0^\infty(\mathbf{R})$, with $|f^{(k)}(x)| \leq 1$ for all $x \in \mathbf{R}$. Expand the integral

$$\int_{\mathbf{R}} \delta_\epsilon(x) f(x) dx$$

in ϵ (Hint: use the Taylor expansion for f around $x = 0$).

(b) Compute the limit $\epsilon \to 0$. The limit will be denoted $\delta(x)$.

(c) Use answers in (a) and (b), and the Fourier transform table, to compute the Fourier transform of $\delta_\epsilon(x)$, and show that $\lim_{\epsilon \to 0} \widehat{\delta_\epsilon} = \frac{1}{2\pi}$.

4. Given the differential operators on a domain $\Omega \subset \mathbf{R}^2$;

$$\begin{aligned} L(u) &= u_{x_1 x_1} + u_{x_2 x_2} - u_{x_2} - 3u, \quad x \in \Omega \\ B(u) &= u + \frac{\partial u}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega, \end{aligned}$$

where $\mathbf{n} = (n_1, n_2)$ is the outward pointing normal on $\partial\Omega$.

Compute the expressions for L^* and B^* (Hint: use integration by parts $\int_\Omega \frac{\partial u}{\partial x_i} v = \oint_{\partial\Omega} uvn_i - \int_\Omega \frac{\partial v}{\partial x_i} u$).

5. (a) Calculate the general solution of

$$u_x + u_y + u = 0$$

(Hint: substitute $v = e^{\frac{1}{2}(x+y)} u$).

(b) Find the unique solution that satisfies the initial condition $u(0,y) = e^y$.

6. (a) Calculate the Fourier series of $|x|$ on $[-\pi, \pi]$.

(b) Compute the sine Fourier series of $|x|$ on $[0, \pi]$.

(c) Explain the different rates of convergence of the Fourier series in (a) and (b).

The cosine Fourier series for f are $\sum_{n=0}^\infty a_n \cos(nx)$, $a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos(nx) dx$, $n \geq 1$, and $a_0 = \frac{1}{\pi} \int_o^\pi f(x) dx$.

The sine Fourier series for f are $\sum_{n=1}^\infty a_n \sin(nx)$, $a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx$, $n \geq 1$.

The Fourier series for f are $\frac{a_0}{2} + \sum_{n=1}^\infty [a_n \cos(nx) + b_n \sin(nx)]$, $a_n = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \cos(nx) dx$, and $b_n = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin(nx)$.

The Fourier transform of f is given by $\mathcal{F}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^\infty f(x) e^{-i\alpha x} dx$, and the Laplace transform by $\hat{f}(s) = \int_0^\infty f(x) e^{-sx} dx$.

Good luck

Table 6-1. Properties of the Fourier Transform

$f(x)$	$F(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx$
1. $f^{(n)}(x)$	$(i\alpha)^n F(\alpha)$
2. $x^n f(x)$	$i^n F^{(n)}(\alpha)$
3. $f(x-c)$	$e^{-ic\alpha}F(\alpha)$ $\quad (c=\text{const.})$
4. $e^{icx}f(x)$	$F(\alpha-c)$ $\quad (c=\text{const.})$
5. $C_1f_1(x)+C_2f_2(x)$	$C_1F_1(\alpha)+C_2F_2(\alpha)$
6. $f(cx)$	$\lvert c\rvert^{-1}F(\alpha/c)$ $\quad (c=\text{const.})$
7. $F(x)$	$\frac{1}{2\pi}f(-\alpha)$
8. $f*g(x)\equiv\int_{-\infty}^{\infty} f(x-y)g(y)\,dy$	$2\pi F(\alpha)G(\alpha)$

Table 6-2. Fourier Transform Pairs

$f(x)$	$F(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx$
1. e^{-cx^2}	$(4\pi c)^{-1/2}e^{-\alpha^2/4c}$ $\quad (c>0)$
2. $e^{-\lambda\lvert x\rvert}$	$\frac{\lambda/\pi}{\alpha^2+\lambda^2}$ $\quad (\lambda>0)$
3. $\frac{2\lambda}{x^2+\lambda^2}$	$e^{-\lambda\lvert\alpha\rvert}$ $\quad (\lambda>0)$
4. $I_A(x)\equiv\begin{cases}1 & \lvert x\rvert<A\\ 0 & \lvert x\rvert>A\end{cases}$	$\frac{\sin A\alpha}{\pi\alpha}$
5. $\frac{2\sin Ax}{x}$	$I_A(\alpha)$
6. $E_a(x)\equiv\begin{cases}0 & x<0\\ e^{-ax} & x>0\end{cases}$	$\frac{1}{2\pi}\frac{1}{a+i\alpha}$ $\quad (\operatorname{Re} a>0)$

The Fourier transform, as described here, applies to functions $f(x)$ in $L^2(-\infty,\infty)$. A related integral transform, called the *Laplace transform*, is defined by

$$\mathscr{L}\{f(t)\}\equiv\int_0^{\infty} f(t)e^{-st}\,dt\equiv\hat{f}(s) \tag{6.18}$$

This transform may be applied to functions $f(t)$ which are defined for $-\infty<t<\infty$ and satisfy $f(t)=0$