
Make up exam**Introduction Partial Differential Equations
for students Mathematics and Physics**Afdeling Wiskunde
Faculteit der Exacte Wetenschappen
Vrije Universiteit Amsterdam

Date: Tuesday February 8, 18:30 - 21:30 (3 hours)

Instructions: 5 problems; *motivate all answers*.No calculators, no books, no formula sheets.

1. (a) Calculate Fourier transform of the following functions

$$f(x) = 2e^{-x^2+4x}, \quad g(x) = e^{-2x}, \quad 1 < x < 4, \quad \text{and} \quad g(x) = 0 \quad \text{eslewhere.}$$

- (b) Compute, using the definition, the Laplace transform of

$$g(x) = x \sin(2x).$$

- (c) Find the inverse Fourier transform

$$\mathcal{F}^{-1}\left(\frac{2}{\alpha^2 - 2\alpha + 2}\right).$$

2. Consider the fourth order equation

$$u_t = u_{xx} + u, \quad x \in [0, \pi], \quad t > 0$$

with initial condition $u(0, x) = f(x) \in C^0([0, \pi])$ and boundary conditions $u(t, 0) = 0$,
and $u(t, \pi) = 0$.

- (a) Give the general solution using separation of variables.
 - (b) Derive the formulas for the Fourier coefficients in terms of the initial function f .
 - (c) Let $f(x) = 1$ on $[0, \pi]$. Compute the solution $u(t, x)$.
 - (d) Prove that $\lim_{t \rightarrow \infty} u(t, x) = \frac{4}{\pi} \sin(x)$ uniformly in $x \in [0, \pi]$.
3. Consider the elliptic equation on a domain
- $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} Lu &= \Delta u + 2u = f, & \mathbf{x} \in \Omega, \\ Bu &= u + \frac{\partial u}{\partial n} = g, & \mathbf{x} \in \partial\Omega, \end{aligned}$$

where $\mathbf{n}$ the outward unit normal on $\partial\Omega$, and $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$.

- (a) Show that the adjoint differential operator
- L^*
- is given by

$$L^*\phi = \Delta\phi + 2\phi$$

by establishing the identity

$$\int_{\Omega} [\phi L\psi - \psi L^*\phi] d_{\Omega}\mathbf{x} = 0,$$

for all $\phi, \psi \in C_0^\infty(\Omega)$. Recall that $C_0^\infty(\Omega)$ are infinitely smooth functions whose support is strictly contained in Ω ; no boundary contributions.

(b) Derive the Green's identity

$$\int_{\Omega} [vLu - uL^*v] d_{\Omega}\mathbf{x} = \dots$$

(c) Find the adjoint boundary operator B^* .

(d) Formulate the differential equation for the Green's function $G(\mathbf{x}, \mathbf{y})$.

(e) Assuming that the Green's function G exists, write down a representation formula for the solution $u(\mathbf{x})$.

4. (a) Determine the general solution of the following first order equation

$$u_x + yu_y = 0.$$

(b) Find the solution that matches the the initial condition $u(0, y) = \sin(y)$.

5. (a) Calculate the Fourier series of x^2 on $[-\pi, \pi]$.

(b) Compute the sine Fourier series of the function $f(x) = 1$ on $[0, \pi]$.

The cosine Fourier series for f are $\sum_{n=0}^{\infty} a_n \cos(nx)$, $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$, $n \geq 1$, and $a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx$.

The sine Fourier series for f are $\sum_{n=1}^{\infty} a_n \sin(nx)$, $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$, $n \geq 1$.

The Fourier series for f are $\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$, and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$.

The Fourier transform of f is given by $\mathcal{F}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$, and the Laplace transform by $\hat{f}(s) = \int_0^{\infty} f(x) e^{-sx} dx$.

Good luck

Table 6-1. Properties of the Fourier Transform

$f(x)$	$F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$
1. $f^{(n)}(x)$	$(i\alpha)^n F(\alpha)$
2. $x^n f(x)$	$i^n F^{(n)}(\alpha)$
3. $f(x-c)$	$e^{-i\alpha c} F(\alpha) \quad (c = \text{const.})$
4. $e^{icx} f(x)$	$F(\alpha - c) \quad (c = \text{const.})$
5. $C_1 f_1(x) + C_2 f_2(x)$	$C_1 F_1(\alpha) + C_2 F_2(\alpha)$
6. $f(cx)$	$ c ^{-1} F(\alpha/c) \quad (c = \text{const.})$
7. $F(x)$	$\frac{1}{2\pi} f(-\alpha)$
8. $f * g(x) \equiv \int_{-\infty}^{\infty} f(x-y) g(y) dy$	$2\pi F(\alpha) G(\alpha)$

Table 6-2. Fourier Transform Pairs

$f(x)$	$F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$
1. e^{-cx^2}	$(4\pi c)^{-1/2} e^{-\alpha^2/4c} \quad (c > 0)$
2. $e^{-\lambda x }$	$\frac{\lambda/\pi}{\alpha^2 + \lambda^2} \quad (\lambda > 0)$
3. $\frac{2\lambda}{x^2 + \lambda^2}$	$e^{-\lambda \alpha } \quad (\lambda > 0)$
4. $I_A(x) \equiv \begin{cases} 1 & x < A \\ 0 & x > A \end{cases}$	$\frac{\sin A\alpha}{\pi\alpha}$
5. $\frac{2 \sin Ax}{x}$	$I_A(\alpha)$
6. $E_a(x) \equiv \begin{cases} 0 & x < 0 \\ e^{-ax} & x > 0 \end{cases}$	$\frac{1}{2\pi} \frac{1}{a + i\alpha} \quad (\text{Re } a > 0)$

The Fourier transform, as described here, applies to functions $f(x)$ in $L^2(-\infty, \infty)$. A related integral transform, called the *Laplace transform*, is defined by

$$\mathcal{L}\{f(t)\} \equiv \int_0^{\infty} f(t) e^{-st} dt \equiv \hat{f}(s) \quad (6.18)$$

This transform may be applied to functions $f(t)$ which are defined for $-\infty < t < \infty$ and satisfy $f(t) = 0$

Table 6-3. Properties of the Laplace Transform

$f(t)$	$\hat{f}(s) = \int_0^{\infty} f(t) e^{-st} dt$
1. $C_1 f_1(t) + C_2 f_2(t)$	$C_1 \hat{f}_1(s) + C_2 \hat{f}_2(s)$
2. $f(at)$	$a^{-1} \hat{f}(s/a) \quad (a > 0)$
3. $f^{(n)}(t)$	$s^n \hat{f}(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$ ($n = 1, 2, \dots$)
4. $t^n f(t)$	$(-1)^n \hat{f}^{(n)}(s) \quad (n = 1, 2, \dots)$
5. $e^{ct} f(t)$	$\hat{f}(s - c) \quad (c = \text{const.})$
6. $H(t - b)f(t - b)$, where $H(t) \equiv \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$	$e^{-bs} \hat{f}(s) \quad (b > 0)$
7. $f * g(t) \equiv \int_0^t f(t - \tau) g(\tau) d\tau$	$\hat{f}(s) \hat{g}(s)$

Table 6-4. Laplace Transform Pairs

$f(t)$	$\hat{f}(s) = \int_0^{\infty} f(t) e^{-st} dt$
1. 1	$\frac{1}{s}$
2. t^n	$\frac{n!}{s^{n+1}} \quad (n = 1, 2, \dots)$
3. e^{kt}	$\frac{1}{s - k}$
4. $\sin at$	$\frac{a}{s^2 + a^2}$
5. $\cos at$	$\frac{s}{s^2 + a^2}$
6. $\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$
7. $\frac{1}{\sqrt{\pi t}} e^{-k^2/4t}$	$\frac{1}{\sqrt{s}} e^{-k\sqrt{s}} \quad (k > 0)$
8. $\frac{k}{\sqrt{4\pi t^3}} e^{-k^2/4t}$	$e^{-k\sqrt{s}} \quad (k > 0)$
9. $\text{erfc}(k/2\sqrt{t})$, where $\text{erfc } z \equiv \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-u^2} du$	$\frac{1}{s} e^{-k\sqrt{s}} \quad (k > 0)$