Midterm/Final exam
Introduction Partial Differential Equations
for students Mathematics and Physics

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Date: Monday December 20 2004, 9:30– 11:30/12:30 (2/3 hours)
Instructions: 3/5 problems; *motivate all answers.*
No calculators, no books, no formula sheets.
For midterm exam 2: problems 1 through 3, total time 2 hours.
For the final exam: problems 1 through 5, total time 3 hours.

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1. (a) Calculate the Fourier transform of the following functions

$$f(x) = \frac{1}{x^2 - 2x + 2}, \quad g(x) = (x^2 + x)e^{-x^2}.$$

   (b) Compute, using the definition, the Laplace transform of

$$g(x) = \sin(x).$$

   (c) Use the convolution formula for the Fourier transform to find

$$\mathcal{F}^{-1}\Big(\frac{\mathcal{F}(\alpha)}{\alpha^2 + 8\alpha + 20}\Big).$$

2. Consider the fourth order equation

$$u_t = -u_{xxxx}, \quad x \in [0, \pi], \quad t > 0$$

with initial condition $u(0,x) = f(x) \in C^0([0,\pi])$ and boundary conditions $u(t,0) = u_{xx}(t,0) = 0$, and $u(t,\pi) = u_{xx}(t,\pi) = 0$.

   (a) Give the general solution using separation of variables.
   (b) Derive the formulas for the Fourier coefficients in terms of the initial function f.
   (c) Let $f(x) = x$ on $[0,\pi]$. Compute the solution $u(t,x)$.
   (d) Prove that $\lim_{t\to\infty} u(t,x) = 0$ uniformly in $x \in [0,\pi]$.

3. Consider the elliptic equation on a domain $\Omega \subset \mathbb{R}^2$

$$Lu = \Delta u + \mathbf{x}\cdot\nabla u = f, \qquad \mathbf{x} \in \Omega,$$
$$Bu = \frac{\partial u}{\partial n} = g, \qquad \mathbf{x} \in \partial\Omega,$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{n}$ the outward unit normal on $\partial\Omega$, and $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$.

(a) Show that the adjoint differential operator L^* is given by

$$L^*\phi = \Delta\phi - \mathbf{x}\cdot\nabla\phi - 2\phi,$$

by establishing the identity

$$\int_\Omega [\phi L\psi - \psi L^*\phi] d_\Omega \mathbf{x} = 0,$$

for all $\phi, \psi \in C_0^\infty(\Omega)$. Recall that $C_0^\infty(\Omega)$ are infinitely smooth functions whose support is strictly contained in Ω; no boundary contributions. (Hint: use the fact that $\mathrm{div}(\phi\psi\mathbf{x}) = 2\phi\psi + \phi\mathbf{x}\cdot\nabla\psi + \psi\mathbf{x}\cdot\nabla\phi$.)

(b) Derive the Green's idenity

$$\int_\Omega [vLu - uL^*v] d_\Omega \mathbf{x} = ...$$

(c) Find the adjoint boundary operator B^*.

(d) Formulate the differential equation for the Green's function $G(\mathbf{x}, \mathbf{y})$.

(e) Assuming that the Green's function G exists, write down a representation formula for the solution $u(\mathbf{x})$.

4. (a) Determine the general solution of the following first order equation

$$yu_x + xu_y = 0.$$

(b) Find the solution that matches the the initial condition $u(0, y) = \frac{1}{1+y^4}$.

5. (a) Calculate the Fourier series of $|x|$ on $[-\pi, \pi]$.

(b) Compute the sine Fourier series of the function $f(x) = x$ on $[0, \pi]$.

(c) Explain the different rates of convergence of the Fourier series in (a) and (b).

The cosine Fourier series for f are $\sum_{n=0}^{\infty} a_n \cos(nx)$, $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)dx$, $n \geq 1$, and $a_0 = \frac{1}{\pi}\int_o^\pi f(x)dx$.

The sine Fourier series for f are $\sum_{n=1}^{\infty} a_n \sin(nx)$, $a_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx$, $n \geq 1$.

The Fourier series for f are $\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$, and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)$.

The Fourier transform of f is given by $\mathcal{F}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx$, and the Laplace transform by $\hat{f}(s) = \int_0^\infty f(x)e^{-sx}dx$.

Good luck

Table 6-1. Properties of the Fourier Transform

$f(x)$	$F(\alpha)=\dfrac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-i\alpha x}\,dx$
1. $f^{(n)}(x)$	$(i\alpha)^n F(\alpha)$
2. $x^n f(x)$	$i^n F^{(n)}(\alpha)$
3. $f(x-c)$	$e^{-ic\alpha}F(\alpha)$ $(c=\text{const.})$
4. $e^{icx}f(x)$	$F(\alpha-c)$ $(c=\text{const.})$
5. $C_1f_1(x)+C_2f_2(x)$	$C_1F_1(\alpha)+C_2F_2(\alpha)$
6. $f(cx)$	$\lvert c\rvert^{-1}F(\alpha/c)$ $(c=\text{const.})$
7. $F(x)$	$\dfrac{1}{2\pi}f(-\alpha)$
8. $f*g(x)\equiv\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$	$2\pi F(\alpha)\,G(\alpha)$

Table 6-2. Fourier Transform Pairs

$f(x)$	$F(\alpha)=\dfrac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-i\alpha x}\,dx$
1. e^{-cx^2}	$(4\pi c)^{-1/2}e^{-\alpha^2/4c}$ $(c>0)$
2. $e^{-\lambda\lvert x\rvert}$	$\dfrac{\lambda/\pi}{\alpha^2+\lambda^2}$ $(\lambda>0)$
3. $\dfrac{2\lambda}{x^2+\lambda^2}$	$e^{-\lambda\lvert\alpha\rvert}$ $(\lambda>0)$
4. $I_A(x)\equiv\begin{cases}1 & \lvert x\rvert<A\\ 0 & \lvert x\rvert>A\end{cases}$	$\dfrac{\sin A\alpha}{\pi\alpha}$
5. $\dfrac{2\sin Ax}{x}$	$I_A(\alpha)$
6. $E_a(x)\equiv\begin{cases}0 & x<0\\ e^{-ax} & x>0\end{cases}$	$\dfrac{1}{2\pi}\dfrac{1}{a+i\alpha}$ $(\text{Re}\,a>0)$

The Fourier transform, as described here, applies to functions $f(x)$ in $L^2(-\infty,\infty)$. A related integral transform, called the *Laplace transform*, is defined by

$$\mathscr{L}\{f(t)\}\equiv\int_0^{\infty} f(t)\,e^{-st}\,dt\equiv\hat{f}(s) \tag{6.18}$$

This transform may be applied to functions $f(t)$ which are defined for $-\infty<t<\infty$ and satisfy $f(t)=0$

Table 6-3. Properties of the Laplace Transform

$f(t)$	$\hat{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt$
1. $C_1 f_1(t) + C_2 f_2(t)$	$C_1 \hat{f}_1(s) + C_2 \hat{f}_2(s)$
2. $f(at)$	$a^{-1}\hat{f}(s/a) \qquad (a > 0)$
3. $f^{(n)}(t)$	$s^n \hat{f}(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$ $(n = 1, 2, \ldots)$
4. $t^n f(t)$	$(-1)^n \hat{f}^{(n)}(s) \qquad (n = 1, 2, \ldots)$
5. $e^{ct} f(t)$	$\hat{f}(s - c) \qquad (c = \text{const.})$
6. $H(t-b) f(t-b)$, where $H(t) \equiv \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$	$e^{-bs}\hat{f}(s) \qquad (b > 0)$
7. $f * g(t) \equiv \int_0^t f(t-\tau)\, g(\tau)\, d\tau$	$\hat{f}(s)\hat{g}(s)$

Table 6-4. Laplace Transform Pairs

$f(t)$	$\hat{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt$
1. 1	$\dfrac{1}{s}$
2. t^n	$\dfrac{n!}{s^{n+1}} \qquad (n = 1, 2, \ldots)$
3. e^{kt}	$\dfrac{1}{s-k}$
4. $\sin at$	$\dfrac{a}{s^2 + a^2}$
5. $\cos at$	$\dfrac{s}{s^2 + a^2}$
6. $\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$
7. $\dfrac{1}{\sqrt{\pi t}} e^{-k^2/4t}$	$\dfrac{1}{\sqrt{s}} e^{-k\sqrt{s}} \qquad (k > 0)$
8. $\dfrac{k}{\sqrt{4\pi t^3}} e^{-k^2/4t}$	$e^{-k\sqrt{s}} \qquad (k > 0)$
9. $\operatorname{erfc}(k/2\sqrt{t})$, where $\operatorname{erfc} z \equiv \dfrac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2}\, du$	$\dfrac{1}{s} e^{-k\sqrt{s}} \qquad (k > 0)$