**Midterm exam** | Afdeling Wiskunde
**Introduction Partial Differential Equations** | Faculteit der Exacte Wetenschappen
**for students Mathematics and Physics** | Vrije Universiteit Amsterdam

Date: Tuesday October 26 2004, 13:30–15:30 (2 hours)
Instructions: 4 problems; *motivate all answers.*
No calculators, no books, no formula sheets.

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1. (a) Determine the type of the following partial differential equations:

$$u_{xx} + 2u_{xy} + u_{yy} + u_y = 0, \quad yu_{xx} - 2u_{xy} + e^x u_{yy} + u = 3.$$

   (b) Show that the equation $5u_{xx} + 4u_{xy} + 4u_{yy} = 0$ is elliptic and find a coordinate transformation to put the equation into canonical form.

2. (a) Determine the general solution of the following first order equation

$$u_x + u_y + u = 0.$$

   (b) Find the solution that matches the the initial condition $u(0, y) = e^y$.

3. Consider the elliptic equation $\Delta u - u_x - u_y = f(x, y)$

   (a) Formulate the maximum principle for the above equation.

   (b) Show that $u(x, y) = e^x + e^y - \frac{1}{2}x - \frac{1}{2}y$ is a solution for $f(x, y) = 1$.

   (c) Let $\Omega = \{(x, y) \ : \ 0 < x < 1, \ \ 0 < y < 1\}$, and consider the above equation with the boundary condition $u = 0$ on $\partial\Omega$, and $f(x, y) = 1$. Use the maximum principle to show that $-1 < u < 1$ on the region $\Omega$ (Hint: Consider the Dirichlet problem for $v = u + x$).

4. Given the function $f(x) = x^2$.

   (a) Calculate the cosine Fourier series of $x^2$ on $[0, \pi]$.

   (b) Determine the full Fourier series of $x^2$ on $[-\pi, \pi]$.

The cosine Fourier series for $f$ are $\sum_{n=0}^{\infty} a_n \cos(nx)$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$, $n \geq 1$, and $a_0 = \frac{1}{\pi}\int_o^{\pi} f(x)dx$.

The Fourier series for $f$ are $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$, and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)$.

*Good luck*