FINAL EXAM FOR DIFFERENTIAL GEOMETRY, FALL 2014

Time: 15:15-18:00 - Books, notes, calculator, etc. are not permitted !
Use for each of the 3 exercises a separate piece of paper !
Do not forget to write your name and student number (UvA/VU) on all papers !
Grading: your grade = 1/3 times your points

Exercise 1

a) How is the differential of a smooth map $f : M \to N$ at a point $p \in M$ defined ? When is f called an immersion ? And is the image $f(M) \subset N$ of an immersion f a submanifold of N ? (3P)

b) Let $\mathbb{R}$ be the real line with its usual manifold structure. Let M denote the manifold which equals $\mathbb{R}$ as a set but with the manifold structure given by the coordinate chart $\phi : \mathbb{R} \to M$, $x \mapsto x^3$. Show that the identity map $\mathbb{R} \to M$ is a homeomorphism, but not a diffeomorphism. Are M and $\mathbb{R}$ diffeomorphic ? (3P)

c) Show that $SL(2) := \{A \in \mathbb{R}^{2\times 2} : \det(A) = 1\}$ is a submanifold of $\mathbb{R}^4$. What is its dimension ? (3P)

Exercise 2

a) What is a Riemannian metric on a smooth manifold M ? Does there always exist a Riemannian metric on any submanifold of $\mathbb{R}^n$? (3P)

b) Give two ways how to define the topology of a manifold M equipped with a Riemannian metric g. (3P)

c) Compute the Lie bracket of the vector fields $X(x) = (-x_2, x_1, 0)$ and $Y(x) = (x_1x_3, x_2x_3, -x_1^2 - x_2^2)$ on $S^2 \subset \mathbb{R}^3$, where (x_1, x_2, x_3) are the coordinates on $\mathbb{R}^3$. What can be said about the flows of X and Y ? (3P)

d) Let $\mathfrak{g}$ be the Lie algebra of a Lie group G and for each $\xi \in \mathfrak{g}$ let ϕ_t^ξ denote the flow of the corresponding left-invariant vector field. Show that the map $\exp : \mathfrak{g} \to G$, $\xi \mapsto \phi_1^\xi(e)$ maps an open neighborhood of 0 in $\mathfrak{g}$ diffeomorphically to an open neighborhood of the neutral element e in G. (3P)

Exercise 3

a) Explain the relation between the exterior derivative and divergence and curl of a vector field on $M = \mathbb{R}^3$. (3P)

b) Compute step by step how a two-form $\omega = f(x_1, x_2)dx_1 \wedge dx_2$ on $\mathbb{R}^2$ changes under a coordinate transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, $(y_1, y_2) \mapsto (x_1, x_2)$. Explain the application to integration on (two-dimensional) manifolds. (3P)

c) Show that the one-form $\omega = (x_1dx_2 - x_2dx_1)/(x_1^2 + x_2^2)$ on $\mathbb{R}^2 \setminus \{0\}$ is closed, but not exact. (3P)