Uitwerking CF 22-06-98

(1) a) $T(z) = \frac{-4}{4} = -1 \qquad T(T(z)) = T(-1) = \frac{2}{1} = 2.$

De cirkel $|z-\tfrac{1}{2}| = \tfrac{3}{2}$ bevat de punten -1 en 2. Het beeld onder T gaat, blijkens (i), ook door -1 en 2. Daar $T(\hat{\mathbb{R}}) = \hat{\mathbb{R}}$ gaat, wegens hoekhouw, de cirkel in zichzelf over. Verder $T(0)=0$. Dus de schijf wordt door T op zichzelf afgebeeld.

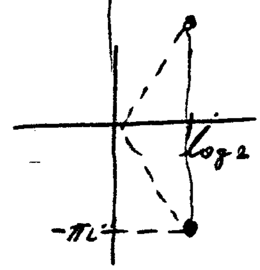
b) $f(z) = \frac{-2e^z}{e^z+2} = \frac{-2}{1+2e^{-z}}.$

Enige singulariteiten: de nulpunten van de noemer. Dit zijn dus polen: $\underline{e^z=-2} = 2e^{\pi i} = e^{\log 2+\pi i}.$

$\Leftrightarrow z = \log 2 + \pi i + 2k\pi i = \underline{\log 2 + (2k+1)\pi i}.$ Het zijn 1e orde nulpunten van e^z+2, dus 1e orde polen van $f(z)$.

$\mathrm{Res}(f, z_k) = \frac{-2}{-2e^{-z}}\Big|_{z_k} = \frac{1}{e^{-z_k}} = e^{z_k} = -2$ (alle k)

De polen z_0 en z_{-1} liggen het dichtst bij 0. De convergentiestraal van de Taylor ontwikkeling is $\sqrt{(\log 2)^2+\pi^2}$.



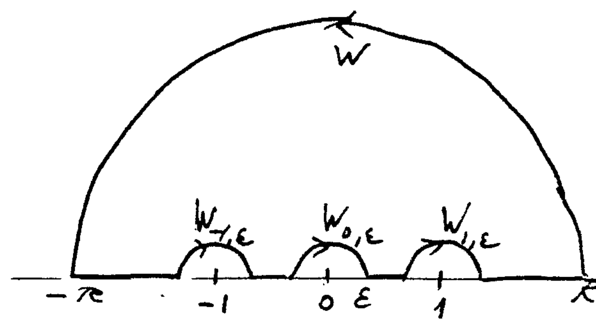
(2) a) $f(z) = \frac{1}{z(z-1)(z+1)} = \frac{1}{z-1}\left\{\frac{1}{z} - \frac{1}{z+1}\right\}$

$\frac{1}{z} = \frac{1}{(z-1)+1} = \frac{1}{1+\frac{1}{z-1}} \cdot \frac{1}{z-1} = \frac{1}{z-1}\sum_{n=0}^{\infty} \frac{(-1)^n}{(z-1)^n} = \sum_{n=-\infty}^{0} (-1)^n (z-1)^n$

$\frac{1}{z+1} = \frac{1}{z-1+2} = \frac{1}{2}\cdot\frac{1}{\frac{z-1}{2}+1} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z-1)^n$

$f(z) = \sum_{n=-\infty}^{0}(-1)^n(z-1)^{n-2} + \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z-1)^{n-1} \qquad (1<|z-1|<2)$

b) Zij W de geschetste kromme.



Dan is $\oint_W g(z)\,dz = 0$ (Cauchy),

waarin
$$g(z) = \frac{z+2}{z(z^2-1)}$$

van de halfcirkelvormige krommen $W_{-1,\varepsilon}$, $W_{0,\varepsilon}$ en $W_{1,\varepsilon}$ geldt, zoals bekend:

$$\lim_{\varepsilon\downarrow 0} \int_{W_{-1,\varepsilon}} g = -\pi i \,\mathrm{Res}(g,-1) = -\pi i \cdot \frac{1}{-1\cdot -2} = \frac{-\pi i}{2}$$

$$\lim_{\varepsilon\downarrow 0} \int_{W_{0,\varepsilon}} g = -\pi i \,\mathrm{Res}(g,0) = -\pi i \cdot \frac{2}{-1} = 2\pi i$$

$$\lim_{\varepsilon\downarrow 0} \int_{W_{1,\varepsilon}} g = -\pi i \,\mathrm{Res}(g,1) = -\pi i \cdot \frac{3}{2} = \frac{-3\pi i}{2}$$

want $\left| \int_{\substack{|z|=R \\ \Im z>0}} g(z)\,dz \right| \le \pi R \cdot \frac{R+2}{R(R^2-1)} \to 0$.

Per saldo: p.v. $\int_{-\infty}^{\infty} (x+2) f(x)\,dx = 0$.

(2) c) We gebruiken het argumentprincipe: Binnen de kromme liggen 3 polen, geen nulpunten. Dus de toename van $\arg f(\gamma(t)) = \underline{2\pi * -3 = -6\pi}$

(3) W a) $x>0$: $\frac{x^2e^x}{1+e^{2x}} \le x^2e^{-x}$ en $\int_0^\infty x^2e^{-x}dx$ conv.

$x<0$: $\frac{x^2e^x}{1+e^{2x}} \le x^2e^{x}$ en $\int_{-\infty}^0 x^2e^{x}dx$ conv.

b) $e^{2z}=-1 \iff 2z = -\pi i \bmod 2\pi i \iff z = \frac{\pi i}{2} \bmod \pi i$

$$\oint_{W_R} f(z)\,dz = 2\pi i \left\{ \mathrm{Res}\left(f, \frac{\pi i}{2}\right)\right\} = \cancel{2}\pi i \cdot \frac{-\frac{\pi^2}{4}\cdot i}{-\cancel{2}} = -\frac{\pi^3}{4}$$

$W_R^{(2)} = [R, R+\pi i]$ c) $$\left| \int_{W_R^{(2)}} f(z)\,dz \right| \le \pi \cdot \frac{(R^2+\pi^2)e^R}{e^{2R}-1} = \frac{\pi(R^2+\pi^2)}{e^R - e^{-R}} \to 0 \quad (R\to\infty)$$

$W_R^{(3)} = [R+\pi i, -R+\pi i]$ d) $$\int_{W_R^{(3)}} \frac{z^2e^z}{1+e^{2z}}\,dz \underset{\substack{z=\pi i+t \\ -R<t<R}}{=} -\int_{-R}^{R} \frac{(\pi i+t)^2 e^{\pi i+t}}{1+e^{2\pi i+2t}}\,dt$$

$$= +\int_{-R}^{R} \frac{(\pi i+t)^2 e^t}{1+e^{2t}}\,dt = \int_{-R}^{R} \frac{-\pi^2+2\pi i t+t^2}{1+e^{2t}} e^t\,dt$$

$$= \int_{-R}^{R} \frac{t^2e^t}{1+e^{2t}}\,dt - \pi^2 \int_{-R}^{R} \frac{e^t}{1+e^{2t}}\,dt + 2\pi i \int_{-R}^{R} \frac{t}{e^{-t}+e^{t}}\,dt$$

De laatste integraal is nul (oneven functie)

$$\int_{-R}^{R} \frac{e^t}{1+e^{2t}}\,dt \underset{e^t=s}{=} \int_{e^{-R}}^{e^R} \frac{1}{1+s^2}\,ds = \arctan s \Big|_{e^{-R}}^{e^R}$$

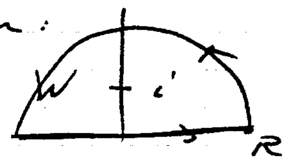
(3) e) $-\frac{\pi^3}{4} = \int_{-R}^{R} \frac{x^2 e^x}{1+e^{2x}}\,dx + \int_{W_R^{(2)}} + \int_{W_R^{(3)}} + \int_{W_R^{4}}$ $\quad W_R^4 = [-R+\pi i, -R]$

neem de limiet $R \to \infty$:

$$-\frac{\pi^3}{4} = 2\int_{-\infty}^{+\infty} \frac{x^2 e^x}{1+e^{2x}}\,dx - \pi^2 \int_{-\infty}^{\infty} \frac{e^t}{1+e^{2t}}\,dt$$

$$\int_{-\infty}^{+\infty} \frac{x^2 e^x}{1+e^{2x}}\,dx = \frac{1}{2}\left\{-\frac{\pi^3}{4} + \pi^2 \cdot \frac{\pi}{2}\right\} = \frac{\pi^3}{8}$$

(4) N a) Bekijk $f(z) = \frac{e^{i\omega z}}{1+z^2}$ en de contour:



$$\oint_W f(z)\,dz = 2\pi i \,\mathrm{Res}(f, i) = 2\pi i \,\frac{e^{-\omega}}{2i} = \frac{\pi}{e^{\omega}}$$

voor $\omega > 0$ is het lemma van Jordan te gebruiken:

$\int_{\substack{|z|=R \\ \mathrm{Im}\, z>0}} f(z)\,dz \to 0 \quad (R \to \infty)$. Dus hebben we:

$\frac{\pi}{e^{\omega}} = \int_{-R}^{R} f(x)\,dx + \int_{\substack{|z|=R \\ \mathrm{Im}\, z>0}} f$. neem $R \to \infty$:

$\int_{-\infty}^{+\infty} \left(e^{i\omega x}/1+x^2\right) dx = \frac{\pi}{e^{\omega}}$. Dus $\int_{-\infty}^{\infty} \frac{\cos \omega x}{1+x^2}\,dx = \frac{\pi}{e^{\omega}}$

voor $\omega < 0$, omdat $\frac{\cos\omega x}{1+x^2}$ even is: $\int_{-\infty}^{\infty} \frac{\cos\omega x}{1+x^2}\,dx = \pi e^{\omega}$

voor $\omega = 0$: $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \arctan x \Big|_{-\infty}^{+\infty} = \pi$.

b) $\int_0^{2\pi} \frac{\sin t}{20+12\sin t}\,dt \;\underset{\frac{dz}{iz}=dt}{\overset{z=e^{it}}{=}}\; \oint_{|z|=1} \frac{\frac{1}{2i}(z-\frac{1}{z})}{20+12(z-\frac{1}{z})/2i}\,\frac{dz}{iz}$

$$= \oint \frac{\frac{1}{2}(z^2-1)}{z(20z + 6i(z^2-1))}\,dz = \frac{-1}{12i}\oint \frac{(z^2-1)\,dz}{z\left(z^2 + \frac{10}{3}iz - 1\right)}$$

$$= \frac{i}{12} \cdot \oint \frac{(z^2-1)\,dz}{z(z+3i)(z+\frac{1}{3}i)} = \frac{i}{12}\cdot 2\pi i \left\{\mathrm{Res}(f,0) + \mathrm{Res}(f,\tfrac{-i}{3})\right\}$$

$\mathrm{Res}(f,0) = \frac{-1}{3i \cdot \frac{1}{3}i} = 1$; $\mathrm{Res}(f,\frac{-i}{3}) = \frac{-10/9}{\frac{-i}{3}\cdot\frac{8i}{3}} = \frac{-10}{9}\cdot\frac{9}{8} = \frac{-5}{4}$

∴ Integraal $= \frac{-2\pi}{12}\cdot\left(1-\frac{5}{4}\right) = \frac{+2\pi}{4\cdot 12} = \frac{\pi}{24}$

⑤ W Zij $\Gamma(s,t) = s\gamma(2\pi t)$ $0 \le s \le 1,\ 0 \le t \le 1$

$= 2s\cos^3(2\pi t) + 2is\sin^3(2\pi t)$

veronderstel, voor zekere s,t is $\Gamma(s,t) = 1+i$

Dan is $\begin{cases} s\cdot\cos^3(2\pi t) = 1/2 \\ s\cdot\sin^3(2\pi t) = 1/2 \end{cases}$

Dus $\cos^2 2\pi t + \sin^2 2\pi t = 2\cdot(1/2)^{2/3}\cdot(1/s)^{2/3}$

Dus $1 = \sqrt[3]{2}\cdot\sqrt[3]{\frac{1}{s^2}}$ of $\sqrt[3]{s^2} = \sqrt[3]{2}$: strijdig met $0 \le s \le 1$

Analoog voor $-(1+i)$.

Zij $f(z) = \frac{\cos z}{z^2-2i}$. De polen van f bevinden zich in $z = \pm\sqrt{2i} = \pm(1+i)$. Dan $\gamma \underset{S}{\sim} 0$, zodat $\oint_\gamma f = 0$.

⑥ N $B(3,1/2) = \dfrac{\Gamma(3)\,\Gamma(1/2)}{\Gamma(7/2)}$

$\Gamma(7/2) = 5/2\,\Gamma(5/2) = 5/2\cdot 3/2\,\Gamma(3/2) = 5/2\cdot 3/2\cdot 1/2\,\Gamma(1/2)$

$\Gamma(3) = 2! = 2$

$\therefore B(3,1/2) = \dfrac{2\cdot\Gamma(1/2)}{\frac{15}{8}\cdot\Gamma(1/2)} = \dfrac{16}{15}.$