

1. a) Let X be a continuous random variable. The probability density function of X is given by

b) Calculate the probability that X is less than 1.

c) Find the value of x such that $P(X \leq x) = 0.5$.

2. Consider a random variable X with the following probability mass function. Find the value of p .

3. Let $X_1, X_2, \dots, X_n$ be independent and identically distributed random variables. Find the probability that $X_1 + X_2 + \dots + X_n \leq 1$.

a) Find the probability that $X_1 + X_2 \leq 1$.

b) Calculate the probability that $X_1 + X_2 + X_3 \leq 1$.

D

4. Consider a random variable X with the following probability mass function. Find the value of p .

Let $X_1, X_2, \dots$ be independent and normally distributed random variables with density p_{θ_0} , where θ_0 denotes the ‘true parameter’. Define, for $\theta \in \mathbb{R}$, the function

$$\psi_{\theta}(x) = (x - \theta)^3 \quad (x \in \mathbb{R})$$

and consider the Z -estimator $\hat{\theta}_n$ defined as a zero of the function

$$\Psi_n(\theta) = \mathbb{P}_n \psi_{\theta} = \frac{1}{n} \sum_{i=1}^n \psi_{\theta}(X_i)$$

- a) Using properties of the function Ψ_n , show that $\hat{\theta}_n$ is well defined. In other words: show that Ψ_n has exactly one point where it becomes zero.
- b) Prove that $\hat{\theta}_n \xrightarrow{P} \mu$.

You may use that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^6 e^{-x^2/2} dx = 15$.

- c) The random variables $\sqrt{n}(\hat{\theta}_n - \theta_0)$ are asymptotically normally distributed. Use the parameter you expect for the asymptotic variance to compute the asymptotic relative efficiency of $\hat{\theta}_n$ and the sample mean $\bar{X}_n$.
5. a) Given a sample $X_1, \dots, X_n$ from a distribution with probability density f , give the definition of the kernel estimator $\hat{f}_{n,h}$ based on this sample and show that the Mean Squared Error (MSE) of $\hat{f}_{n,h}(x)$ decomposes in a bias- and variance term.
- b) Formulate the Glivenko Cantelli theorem and give an outline of its proof (line of thought, basic ingredients).

1a	1b	1c	2	3a	3b	4a	4b	4c	5a	5b
4	3	3	3	4	3	2	2	3	3	2

*The final grade is (number of points + 4)/3.6
Good luck with the exam!*