Exam Asymptotic Statistics

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1. a) Give the definition of (uniform) tightness of a sequence of random variables and show that the sequence (X_n), with X_n uniformly distributed on $[n, n+1]$, is not uniformly tight.

   b) Give an example of a sequence of random variables X_n such that X_n converges in distribution, but not in probability.

   c) Let $X_1, X_2, \dots$ be a sequence of random variables such that for all $\epsilon > 0$,

   $$\sum_{n=1}^{\infty} P(|X_n| > \epsilon) < \infty.$$

   Prove that $X_n \to 0$ almost surely.

   d) Let $U_1, U_2, \dots$ i.i.d. random variables, uniformly distributed on $[0,1]$. Define $Y_n = \max_{1 \le i \le n} U_i$. Prove that $Y_n \to 1$ almost surely. **Hint:** use part c) of this exercise, with $X_n = Y_n - 1$.

2. Let, for $n = 1, 2, \dots$, $X_n \sim \text{Bin}(n, p_1)$ and $Y_n \sim \text{Bin}(n, p_2)$ be independent random variables. We wish to test the hypothesis $H_0 : p_1 = p_2 = a$ for some fixed $a \epsilon (0,1)$, using the test statistic

   $$C_n = \frac{(X_n - na)^2 + (Y_n - na)^2}{na(1-a)}.$$

   Derive the asymptotic distribution under H_0 for C_n as $n \to \infty$.

3. Let $X_1, X_2, \dots$ be i.i.d. random variables with density $f(x) = \lambda \exp(-\lambda x)$, $x \ge 0$, for some unknown parameter $\lambda > 0$. Let $\bar{X}_n$ be the sample mean of the first n random variables, $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. The maximum likelihood estimator for the parameter λ, based on $X_1, \dots, X_n$, is given by $\hat{\lambda}_n = 1/\bar{X}_n$. It is given that $EX_i = \lambda^{-1}$ and $\text{Var}X_i = \lambda^{-2}$.

   a) Show that $\sqrt{n}(\hat{\lambda}_n - \lambda) \rightsquigarrow N(0, \lambda^2)$.

   b) We wish to estimate $P(X_i > 1) = e^{-\lambda}$. Inspired by a), we use the estimator $S_n = e^{-\hat{\lambda}_n}$. Show that this estimator is consistent and

   $$\sqrt{n}(S_n - e^{-\lambda}) \rightsquigarrow N(0, \lambda^2 e^{-2\lambda}).$$

   c) Another estimator that can be used to estimate $P(X_i > 1) = e^{-\lambda}$, is

   $$T_n = \frac{1}{n}\sum_{i=1}^n 1_{\{X_i > 1\}} = \#\{i = 1, \dots, n : X_i > 1\}/n.$$

   Prove that

   $$\sqrt{n}(T_n - e^{-\lambda}) \rightsquigarrow N(0, e^{-\lambda}(1 - e^{-\lambda})).$$

   d) Compute the Asymptotic Relative Efficiency of S_n relative to T_n and give your ideas on which estimator of the two to prefer.

4. Let $X_1, \ldots, X_n$ be independent $N(\mu, 1)$ distributed random variables. Define $\hat{\theta}_n$ as the point of minimum of the function $\theta \to \sum_{i=1}^n (X_i - \theta)^4$ or, equivalently, as the zero of the function $\theta \to \sum_{i=1}^n (X_i - \theta)^3$.

   a) Show that $\hat{\theta}_n \xrightarrow{P} \theta_0$ for some θ_0. Which θ_0?

   b) Derive the asymptotic (normal) distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$. You may use that the fourth moment of the standard normal distribution is 15.

   Consider a class of probability densities on $\mathbb{R}$, $\{p_\theta : \theta \epsilon \Theta\}$. Fix $\theta_0 \epsilon \Theta$. The Kullback Leibler divergence of p_θ and p_{θ_0} is defined by

$$M(\theta) = E_{\theta_0} \log \frac{p_\theta(X)}{p_{\theta_0}(X)}.$$

   c) Show that $M(\theta) \leq 0$ for all $\theta \epsilon \Theta$.

5. Let (F_n) be a sequence of monotonically increasing functions on $\mathbb{R}$ and F a continuous, monotonically increasing function on $\mathbb{R}$, with $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$. Suppose that for each $x \epsilon \mathbb{R}$, $F_n(x) \to F(x)$ as $n \to \infty$. Prove that $\sup_{x \epsilon \mathbb{R}} |F_n(x) - F(x)| \to 0$ as $n \to \infty$.

Grading:

1a:2	1c:2	2:3	3b:2	3d:2	4b:4	5:3
1b:2	1d:3	3a:2	3c:2	4a:3	4c:2	

The final grade is computed as follows: $\frac{\text{number of points} + 4}{3.6}$. Good luck with the exam!