Tentamen Algebraïsche Specificaties

29 augustus 2005, 13.30-16.30

You have **three** hours for answering the questions below. In total, you can earn 100 points. Of these, 10 are given to you for free (do not ask me why). Good luck!

Question 1 (2 + 3 + 2 + 2 + 3 + 3 points)
Let A be an arbitrary non-empty set.

(a) Consider the function

$$zip : (A^\omega \times A^\omega) \to A^\omega$$

that is defined, for all $\sigma, \tau \in A^\omega$, by the following differential equation:

derivative	initial value
$zip(\sigma,\tau)' = zip(\tau,\sigma')$	$zip(\sigma,\tau)(0) = \sigma(0)$

Compute the stream $zip(\alpha, \beta)$, for arbitrary streams $\alpha, \beta \in A^\omega$.

(b) Define the function

$$even : A^\omega \to A^\omega, \quad even(\sigma) = (\sigma(0), \sigma(2), \sigma(4), \ldots)$$

by means of a differential equation. Similarly, define the function

$$odd : A^\omega \to A^\omega, \quad odd(\sigma) = (\sigma(1), \sigma(3), \sigma(5), \ldots)$$

by means of a differential equation.

(c) Consider the function

$$Zip : (A^\omega \times A^\omega) \to A^\omega$$

that is defined, for all $\sigma, \tau \in A^\omega$, by the following differential equation:

derivative	initial value
$Zip(\sigma,\tau)' = Zip(\tau',\sigma')$	$Zip(\sigma,\tau)(0) = \sigma(0)$

Compute the stream $Zip(\alpha, \beta)$, for arbitrary streams $\alpha, \beta \in A^\omega$.

(d) Let $R \subseteq A^\omega \times A^\omega$ be a relation on A^ω. When is R a bisimulation relation?

(e) Express the function Zip, from part (c), in terms of the functions from parts (a) and (b).

(f) Prove your statement under (e) by coinduction.

Question 2 (3 + 3 + 3 + 3 + 3 points)
Consider the set $\mathbb{R}^\omega$ of streams of real numbers.

(a) Give the definition of the product $\sigma \times \sigma$ of a stream σ in $\mathbb{R}^\omega$ with itself.

(b) Give a general formula for the derivative $(\sigma \times \sigma)'$ of $\sigma \times \sigma$, for an arbitrary stream $\sigma \in \mathbb{R}^\omega$.

(c) Recall the following definitions:

$$X = (0, 1, 0, 0, 0, \ldots), \quad [r] = (r, 0, 0, 0, \ldots) \ \ (r \in \mathbb{R})$$

Compute the first 9 elements of each of the following streams (as usual, we simply write r instead of $[r]$):

(i) $X^3 \times 3 \times X^3$
(ii) $(7 + 6X^4) + (3 - 8X^3 + X^6)$
(iii) $(2 + X)^4$

(d) Compute the stream $X^2 \times \sigma \times X^3$, for an arbitrary stream $\sigma \in \mathbb{R}^\omega$.

(e) Consider a fixed but unknown stream $\sigma \in \mathbb{R}^\omega$. Find $\tau \in A^\omega$ such that the following equality holds:

$$\sigma = \sigma(0) + (\sigma(1) \times X) + \tau$$

Question 3 (5 + 5 + 5 points)

(a) Consider the following differential equation:

derivative	initial value
$\sigma' = -(X \times \sigma)$	$\sigma(0) = 3$

Compute the stream σ.

(b) Consider the following differential equation:

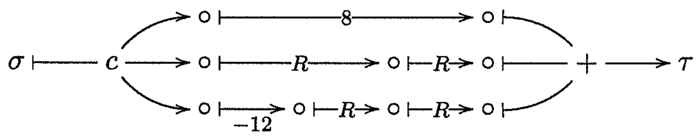
derivative	initial value
$\tau' = X^4 \times \tau$	$\tau(0) = 2$

Compute the stream τ.

(c) Consider the stream $\delta = \frac{2X+X^2}{1-X^3}$. Compute the derivative δ' of δ.

Question 4 (3 + 3 + 4 points)

(a) What is the stream function $f : \mathbb{R}^\omega \to \mathbb{R}^\omega$ that is implemented by the following stream circuit?:

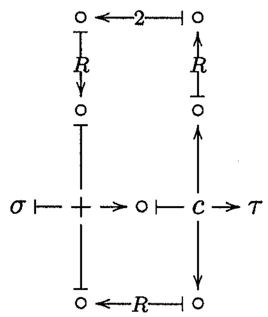


(b) Construct a circuit that implements the same stream function, using only two register gates.

(c) Give a general description of the kind of stream functions that can be implemented by finite so-called feed-forward circuits (i.e., finite circuits without feedback loops) such as the one in part (a).

Question 5 (5 + 10 points)

(a) Compute the stream function $f : \mathbb{R}^\omega \to \mathbb{R}^\omega$ that is implemented by the following circuit:



(b) Does there exist an equivalent circuit (that is, implementing the same stream function) with two registers? If so, show one. If not, explain why not.

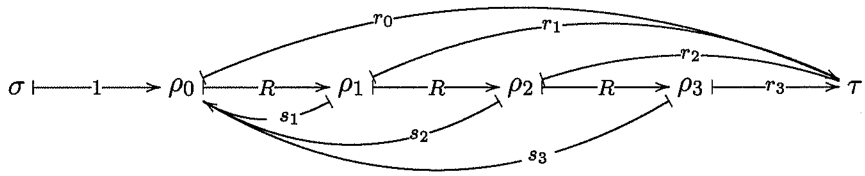
Question 6 (4 + 6 points)
Consider the set $\mathbb{R}^\omega$ of streams of real numbers.

(a) Compute the stream $zip(\frac{1}{1-X^2}, \frac{2}{1-X})$.

(b) Prove that the stream in part (a) is rational.

Question 7 (5 + 5 points)

(a) Consider the following circuit (where we use the convention of not drawing copiers and adders explicitly):



What is the stream function $f : \mathbb{R}^\omega \to \mathbb{R}^\omega$ that is implemented by this circuit, for the case that

$$r_0 = r_1 = 1,\ r_2 = r_3 = 0,\ s_1 = s_3 = 0,\ s_2 = 1$$

(b) Consider all possible circuits that are equivalent to the one of part (a) (that is, implementing the same stream function). How many registers do such circuits contain at least? Explain.