Tentamen Algebraïsche Specificaties

27 mei 2004

You have **three** hours for answering the questions below. In total, you can earn 100 points. Of these, 10 are given to you for free. Good luck!

Question 1 (2 + 3 + 2 + 2 + 3 + 3 points)
Let A be an arbitrary non-empty set.

(a) Consider the function
$$even : A^\omega \to A^\omega$$
that is defined, for all $\sigma \in A^\omega$, by the following differential equation:

derivative	initial value
$even(\sigma)' = even(\sigma'')$	$(even(\sigma))(0) = \sigma(0)$

Compute the stream $even(\sigma)$.

(b) Define the function
$$odd : A^\omega \to A^\omega, \quad odd(\sigma) = (\sigma(1), \sigma(3), \sigma(5), \ldots)$$
by means of a differential equation.

(c) Define the function
$$f : A^\omega \to A^\omega, \quad f(\sigma) = (\sigma(2), \sigma(6), \sigma(10), \ldots)$$
for all $\sigma \in A^\omega$, by means of a differential equation.

(d) Let $R \subseteq A^\omega \times A^\omega$ be a relation on A^ω. When is R a bisimulation relation?

(e) Formulate the coinduction proof principle.

(f) Prove by coinduction, for all $\sigma \in A^\omega$: $f(\sigma) = odd(even(\sigma))$ (with f as defined in part (c)).

Question 2 (3 + 3 + 4 points)
Let A be an arbitrary non-empty set.

(a) Consider the function

$$Z : (A^\omega \times A^\omega) \to A^\omega$$

that is defined, for all $\sigma, \tau \in A^\omega$, by the following differential equation:

derivative	initial value
$Z(\sigma,\tau)'' = Z(\sigma',\tau')$	$Z(\sigma,\tau)(0) = \sigma(0),\ Z(\sigma,\tau)'(0) = \tau(0)$

Compute the stream $Z(\alpha, \beta)$, for arbitrary streams α and β in A^ω.

(b) Define the function

$$D : A^\omega \to A^\omega, \quad D(\sigma) = (\sigma(0), \sigma(0), \sigma(1), \sigma(1), \sigma(2), \sigma(2), \ldots)$$

for all $\sigma \in A^\omega$, by means of a differential equation.

(c) How are the functions Z and D related? Prove your claim by coinduction.

Question 3 (3 + 3 + 3 + 3 + 3 points)
Consider the set $\mathbb{R}^\omega$ of streams of real numbers.

(a) Give the definition of the convolution product $\sigma \times \tau$ of two streams σ and τ in R^ω.

(b) Define the sum $\sigma + \tau$ of two streams σ and τ in R^ω, by means of a differential equation.

(c) Recall the following definitions:

$$X = (0, 1, 0, 0, 0, \ldots), \quad [r] = (r, 0, 0, 0, \ldots) \ (r \in \mathbb{R})$$

Compute the first 7 elements of each of the following streams (as usual, we simply write r instead of $[r]$):

(i) $X \times 3 \times X^3$

(ii) $(3 + 8X^2) + (5 - 7X^2 + X^5)$

(iii) $(1 + 2X)^3$

(d) Compute the stream $X \times \sigma$, for an arbitrary stream $\sigma \in \mathbb{R}^\omega$.

(e) Prove the following equality, for all $\sigma \in \mathbb{R}^\omega$:

$$\sigma = \sigma(0) + (X \times \sigma')$$

Question 4 (3 + 3 + 3 + 3 + 3 points)

(a) Consider the following differential equation:

derivative	initial value
$\sigma' = \sigma$	$\sigma(0) = 1$

Compute the stream σ.

(b) Consider the following differential equation:

derivative	initial value
$\tau' = X \times \tau$	$\tau(0) = 1$

Compute the stream τ.

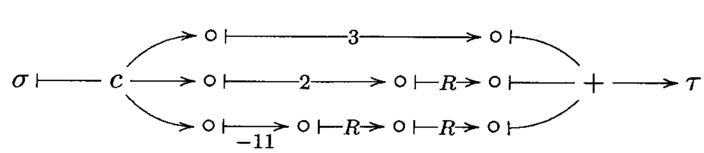
(c) Show that the stream τ (from part (b)) satisfies the following equality: $\tau \times (1 - X^2) = 1$. (In other words: $\tau = \frac{1}{1-X^2}$.)

(d) Consider the stream $\delta = \frac{1}{1+X^3}$. Define δ by means of a differential equation.

(e) Compute the first 9 elements of the stream δ.

Question 5 (4 + 2 + 4 points)

(a) What is the stream function $f : \mathbb{R}^\omega \to \mathbb{R}^\omega$ that is implemented by the following stream circuit?:

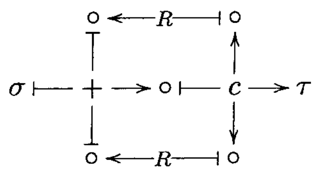


(b) What is the stream generated by this circuit?

(c) Give a general description of the kind of streams that can be generated by finite so-called feed-forward circuits (i.e., finite circuits without feedback loops) such as the one in part (a).

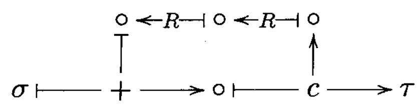
Question 6 (5 + 5 + 5 points)

(a) Compute the stream function $f : \mathbb{R}^\omega \to \mathbb{R}^\omega$ that is implemented by the following circuit:



(b) Does there exist an equivalent circuit (that is, implementing the same stream function) that has only one register? If so, show one. If not, explain why not.

(c) Compute the stream function $g : \mathbb{R}^\omega \to \mathbb{R}^\omega$ that is implemented by the following circuit:



Question 7 (4 + 6 points)
Consider the set $\mathbb{R}^\omega$ of streams of real numbers.

(a) Compute the stream $zip(\frac{1}{1-X}, \frac{2}{1-X})$.

(b) Prove that the stream in part (a) is rational.