Faculty of Sciences	**Algebra II**
VU University Amsterdam	**Exam 26-5-2009 (8:45-11:30)**

Attention

(1) **Attempt all 11 questions.** If you cannot do a part of a problem then you may still use the result in the rest of the problem.

(2) The exam and solutions (in Dutch) will appear shortly on blackboard.

1. Knowledge questions

(1) (a) Define the kernel of a group homomorphism $f : G \to H$ and show that is is a normal subgroup of G.
   (b) Define the *center* $Z(G)$ of a group G.
   (c) Define the *commutator subgroup* $[G, G]$ of a group G.

(2) Let G be a group, X a non-empty set on which G acts (on the left).
   (a) What is meant by this *action* of G on X?
   (b) Define the *orbit* and the *stabilisor* of an element x in X.
   (c) Show that if x and y are in the same orbit in X, then their stabilisors are conjugate.

(3) Let R be a commutative ring with 1 (identity), $f \neq 0$ a polynomial in $R[x]$ with $\deg(f) \geq 1$ and leading coefficient in R^*.
   (a) What does *division with remainder* state in this case?
   (b) Use this to describe the different classes in the quotient ring $R[x]/(f)$. You have to prove your statement.

(4) Let R be an integral domain.
   (a) When is an element x of R with $x \neq 0$
      (i) a *prime element* of R;
      (ii) an *irreducible element* of R?
   (b) When is R
      (i) a Euclidean domain;
      (ii) a principal ideal domain;
      (iii) a unique factorization domain?

2. Problems

Answers without reasoning score poorly so give good arguments everywhere.

(5) Determine which of the four groups A_4, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and $S_3 \times \mathbb{Z}/2\mathbb{Z}$ are mutually isomorphic and which are not.

(6) We consider the subset

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \text{ with } a, b, c \text{ in } \mathbb{R} \text{ and } ac = 1 \right\}$$

of $GL_2(\mathbb{R})$, the set of invertable 2×2-matrices with real coefficients.
   (a) Show that G is a subgroup of $GL_2(\mathbb{R})$ under the usual multiplication of matrices in $GL_2(\mathbb{R})$.
   (b) Show that the map $f : G \to \mathbb{R}^*$ given by $f(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)) = a$ is a homomorphism of groups.
   (c) Determine the commutator subgroup $[G, G]$ of G. Formulate the theorems you use for this. *Hint*: compute the commutators $[x, y]$ with $x = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1/2 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$.

(7) Let $f : G \to G'$ be a homomorphism of groups and H a subgroup of G with $[G : H]$ finite.

(a) Show that $[G : H] \geq [f(G) : f(H)]$.
(b) Show that if $\ker(f) \subseteq H$ then we even have $[G : H] = [f(G) : f(H)]$.

(8) Let $R = \mathbb{Z}[i]$, the ring of Gaussian integers.
(a) Use the Euclidean algorithm on R in order to write the ideal $(2 + 4i, 5 + 5i)$ of R as a principal ideal.
(b) Use the norm in order to determine factorizations of $2 + 3i$ and of $2 + 4i$ into irreducibles of R.

(9) Let x be a variable and $R = \mathbb{Z}[x]$.
(a) Show that the ideal $(5, x^2 + 1)$ is *not* a principal ideal of R.
(b) Factorize $f = 7x^3 + 28x^2 + 42x + 21$ into irreducible factors in R.

(10) **In this problem, also formulate the theorems that you are using in your reasoning.**
Let $R = \mathbb{Z}[\sqrt{-5}]$, I the ideal $(3 - \sqrt{-5}, 7)$ of R, and $f : R \to \mathbb{Z}/7\mathbb{Z}$ the map given by $f(a + b\sqrt{-5}) = \overline{a + 3b}$.
(a) Show that f is a homomorphism of rings.
(b) Show that there is a ring isomorphism $R/I \cong \mathbb{Z}/7\mathbb{Z}$.
(c) Is I a maximal ideal and/or a prime ideal of R?

(11) (a) Use the norm in order to determine all divisors of 4 in the ring $\mathbb{Z}[\sqrt{-3}]$.
(b) Is $\mathbb{Z}[\sqrt{-3}]$ a unique factorization domain?

Points										
1a: 4	2a: 3	3a: 3	4a: 4	5: 6	6a: 3	7a: 3	8a: 4	9a: 5	10a: 2	11a: 3
1b: 2	2b: 4	3b: 4	4b: 6		6b: 2	7b: 4	8b: 4	9b: 4	10b: 4	11b: 3
1c: 2	2c: 3				6c: 4				10c: 4	
Maximum total = 90									Grade = 1 + Total/10	