Vrije Universiteit Amsterdam **29-03-2023 8:30-11:15**
Midterm Topology 2022-2023 **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. [**5+3+3+5+4 Points**] Define the ray topology on $\mathbb{R}$ by

$$\mathcal{T}_{\text{ray}} = \{(a, \infty) \subset \mathbb{R} \,|\, a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

a) Show that $\mathcal{T}_{\text{ray}}$ is a topology on $\mathbb{R}$.

b) Show that $\mathcal{T}_{\text{ray}}$ is not Hausdorff.

c) Give the definition of the convergence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a topological space X.

d) Suppose that $\{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}$ is a sequence that converges to $x \in \mathbb{R}$ in the ray topology. Show that x_n converges to $y \in \mathbb{R}$ in the ray topology for every $y \leq x$.

e) Let the sequence $\{x_n\}_{n\in\mathbb{N}}$ be given by $x_n = n$. Determine if the sequence converges in the ray topology on $\mathbb{R}$. If it converges, determine all points it converges to.

Question 2. [**10 Points**] Let (X, d) be a metric space, and $A \subset X$ be a finite subset. Prove that A is closed in X.

Question 3. [**8+5+7 Points**] Let $S^0 = \{-1, 1\} \subset \mathbb{R}$. Let X be any space.

a) Show that there is a continuous and surjective function $f : X \to S^0$ if and only if X is not connected.

Let

$$\mathrm{GL}(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,|\, a, b, c, d \in \mathbb{R}, \text{ and } ad - bc \neq 0 \right\}$$

be the set of 2×2 real matrices matrices that are invertible. This set has a topology as a subspace of $\mathbb{R}^4$ in the standard topology.

b) Show that $\mathrm{GL}(2)$ is an open subset of $\mathbb{R}^4$.

c) Show that $\mathrm{GL}(2)$ is not connected.

Exam continued on the back $\Rightarrow \ldots$

Question 4. [**3+10 Points**] Let $f : X \to Y$ be a function between topological spaces X and Y and let $g : X \to X \times Y$ be given by

$$g(x) = (x, f(x))$$

a) Show that g is injective.

Recall that g is said to be an imbedding if it is injective, continuous and the restriction $g' : X \to Z$ of g to the range $Z = g(X)$ (equipped with the subspace topology) is a homeomorphism.

b) Show that f is continuous if and only if g is an imbedding.

Question 5. [**9+8 Points**] Let X be a set. Let I be an indexing set, and suppose for each $\alpha \in I$, there is a topological space X_α and a function $f_\alpha : X_\alpha \to X$. Let

$$\mathcal{T} = \{U \subset X \mid f_\alpha^{-1}(U) \text{ is open in } X_\alpha \text{ for each } \alpha \in I\}$$

a) Show that $\mathcal{T}$ is a topology on X.

b) Show that a function $g : X \to Y$ is continuous if and only if $g \circ f_\alpha : X_\alpha \to Y$ is continuous for all $\alpha \in I$.

The topology on X is called the final *topology w.r.t. the maps f_α. It generalizes the quotient topology.*

Question 6. [**10 Points**] Proof the following part of a Theorem from the book.

Theorem (2.17.5 (a))**.** Let X be a topological space and $A \subset X$. Show that $x \in \overline{A}$ if and only if every neighborhood U of x intersects A, i.e. $U \cap A \neq \emptyset$.