Vrije Universiteit Amsterdam **02-06-2022 18:45-21:30**
Final Topology 2021-2022 **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. [**3+7+7 Points**] Let X be a topological space and $A, B \subset X$ such that $A \cup B = X$.

a) Give the definition of a compact space.

b) Suppose that A and B are compact, show that X is compact.

c) Suppose that A and B are compact, and X is Hausdorff. Show that $A \cap B$ is compact.

Question 2. [**5+10 Points**] Let X, Y be non-empty topological spaces, and suppose that Y is compact. Let $x \in X$ and let $V \subset X \times Y$ be an open set such that $\{x\} \times Y \subset V$.

a) Show that for every $y \in Y$ there exists open sets $A_y \subset X$ and $B_y \subset Y$ such that $(x, y) \in A_y \times B_y$ and $A_y \times B_y \subset V$.

b) Show that there exists an open neighborhood $U \subset X$ such that $x \in U$ and $U \times Y \subset V$.

Question 3. [**3+10 Points**] Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the circle,

$$GL_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc > 0 \right\}$$

be the 2×2 matrices with positive determinant, and

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

the 2×2 matrices with determinant one.

a) Give the definition of a deformation retract.

b) Show that $SL_2(\mathbb{R})$ is a deformation retract of $GL_2^+(\mathbb{R})$.

Question 4. [**3+12 Points**] Recall the definitions in Question 3. Define the maps $i : S^1 \to GL_2^+(\mathbb{R})$ by $i(x, y) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ and $r : GL_2^+(R) \to S^1$ by $r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{1}{\sqrt{a^2+c^2}}(a, c)$.

a) Discuss why r is well-defined.

b) Show that

$$i_* : \pi_1(S^1, (1, 0)) \to \pi_1\left(GL_2^+(\mathbb{R}), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is injective and

$$r_* : \pi_1\left(GL_2^+(\mathbb{R}), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \to \pi_1(S^1, (1, 0))$$

is surjective.

Question 5. [**3+7 Points**] Let $X = \{(x, y, z) \in \mathbb{R}^3 \,|\, (x, y) \neq (0, 0)\}$.

a) Sketch the space X.

b) Compute the fundamental group $\pi_1(X, (1, 0, 0))$.

Hint: X deformation retracts to a familiar space.

Question 6. [**5+5+5+5 Points**] Let $p : E \to B$ be a covering map, and $e \in E$ and $b = p(e)$.

a) Recall how the lifting correspondence

$$\phi : \pi_1(B, b) \to p^{-1}(b)$$

is defined. *You do not have to show that the lifting correspondence is well-defined.*

b) Show that ϕ is surjective if E is path-connected

In part c) and d) assume that E is path-connected and B is simply connected.

c) Show that p is injective.

d) Show that p is a homeomorphism.