Vrije Universiteit Amsterdam **29-03-2022 18:45-21:30**
Midterm Topology 2021-2022 **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. [**15 Points**] Let (X, d) a metric space and $A \subset X$ a subset. Define the function $d_A : X \to [0, \infty)$ by

$$d_A(y) = \inf_{x \in A} d(x, y).$$

Show that $d_A(y) = 0$ if and only if $y \in \overline{A}$.

Question 2. [**10 Points**] Let X be a topological space and $f, g : X \to \mathbb{R}$ be two continuous functions, where $\mathbb{R}$ is equipped with the standard topology. Show that the set

$$U = \{x \in X \mid f(x) < g(x)\}$$

is an open set in X.

Question 3. [**5+5+5+5 Points**] Consider $\mathbb{R}$ with the standard topology. Let

$$\mathbb{N} = \{1, 2, \ldots\} \subset \mathbb{R} \quad \text{and} \quad B = \left\{ \frac{1}{n} \in \mathbb{R} \,|\, n \in \mathbb{N} \right\} \cup \{0\}.$$

The sets $\mathbb{N}$ and B are topologized as subspaces of $\mathbb{R}$. The spaces $\mathbb{N}$ and B have the same cardinality, but are not homeomorphic which you are going to show below.

a) Show that any converging sequence $x_n \in \mathbb{N}$ must eventually be constant. *Hint: Show that* $\mathbb{N}$ *has the discrete topology.*

b) Give a converging sequence $x_n \in B$ that is not eventually constant and prove that the sequence converges.

c) Conclude that $\mathbb{N}$ and B are not homeomorphic.

d) Show that $\mathbb{N}$ is homeomorphic to $B \setminus \{0\}$. *You may use continuity of standard functions on* $\mathbb{R}$ *without proving that they are continuous.*

Question 4. [**15 Points**] Prove (part of) Theorem 22.2 from the book:

Theorem (2.22). *Let* $p : X \to Y$ *be a quotient map. Let* Z *be a space and let* $g : X \to Z$ *be a map that is constant on each set* $p^{-1}(\{y\})$ *for* $y \in Y$*. Then* g *induces a map* $f : Y \to Z$ *such that* $f \circ p = g$*. The induced map* f *is continuous if and only if* g *is continuous.*

Exam continues on the back ⇒

Question 5. [**3+6+6 Points**] Let X be a space with at least two points. A space Y is called X-connected if for every $x, y \in Y$ there exists a continuous map $f : X \to Y$ with $x, y \in f(X)$.

a) Give the definition of a connected space.

b) Suppose that X is connected and Y is X-connected. Show that Y is connected.

c) Suppose that X is not connected and Y is any non-empty space. Show that Y is X-connected.

Question 6. [**3+12 Points**] Let X be a topological space and Y a Hausdorff space. Let $f : X \to Y$ be a continuous map.

a) Give the definition of a Hausdorff space.

b) Show that the graph
$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\}$$
is closed.