Vrije Universiteit Amsterdam **28-06-2021 15:30-18:15**
Resit Topology **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. [**2+8 Points**] Let $Y = [-2,2] \times [-2,2]$ equipped with the usual subspace topology of $\mathbb{R}^2$. Consider

$$\begin{aligned} \mathbf{8} &= \{(x,y) \in Y \mid x^2 + (y-1)^2 = 1 \quad \text{or} \quad x^2 + (y+1)^2 = 1\}, \\ X &= \{(x,y) \in Y \mid x = \pm y\}, \\ O &= \{(x,y) \in Y \mid x^2 + y^2 = 1\}. \end{aligned}$$

a) Sketch the spaces $\mathbf{8}, X, O$.

b) Prove that $\mathbf{8}, X, O$ are all not homeomorphic to each other.

Question 2. [**15 Points**] Prove Theorem 26.6 from the book: Let $f : X \to Y$ be a continuous and bijective function. Suppose that X is compact and Y is Hausdorff. Then f is a homeomorphism.

Question 3. [**5+5+5+5 Points**] Let (X, d) be a metric space and $A \subset X$ be a non-empty subset. For every $x \in X$ we define the *distance from* A by

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

a) Show that $d_A : X \to \mathbb{R}$ given by $d_A(x) = d(x, A)$ is a continuous function.

b) Assume that A is closed and let $x \in X \setminus A$. Show that $d(x, A) > 0$.

c) Let $A, B \subset X$ are disjoint where A is closed and B is compact. Show that

$$d(A, B) = \inf\{d(x, y) \mid x \in A \quad \text{and} \quad y \in B\} > 0.$$

d) Give disjoint $A, B \subset \mathbb{R}^2$ such that A, B are closed but $d(A, B) = 0$.

Question 4. [**3+6+6 Points**] Let $p : E \to B$ be a covering map. And let $e \in E$ and $b = f(e)$.

a) Give the definition of the closure $\overline{A}$ of a subset $A \subset B$.

b) Let $A \subset B$ such that $b \in \overline{A}$. Show that $e \in \overline{p^{-1}(A)}$

c) Give an example of a continuous map $f : X \to Y$, $x \in X$, $y \in Y$, and $Z \subset Y$ such that $f(x) = y$ and $y \in \overline{Z}$ but $x \notin \overline{f^{-1}(Z)}$.

Question 5. [**7+8 Points**] Let $A \subset B \subset C$ be spaces. Recall that B is said to be a deformation retract of C if there exists a map $H : C \times [0,1] \to C$ such that $H(x, 0) = x$ and $H(x, 1) \in B$ for all $x \in C$ and that $H(b, t) = b$ for all $b \in B$ and $t \in [0, 1]$.

a) Let $f : X \to C$ be a continuous map. Suppose that B is a deformation retract of C. Show that f is homotopic to a map $g : X \to C$ such that $g(X) \subset B$.

b) Suppose A is a deformation retract of B and B is a deformation retract of C. Show that A is a deformation retract of C.

Question 6. [**6+3+6 Points**] Let $n \geq 2$. Recall that $\mathbb{RP}^n = S^n/\sim$ where $x \sim y$ iff $x = \pm y$. We have seen that $\pi_1(\mathbb{RP}^n, [x]) \cong \mathbb{Z}/2\mathbb{Z}$.

a) Show that for any map $f : \mathbb{RP}^n \to S^1$ the induced map $f_* : \pi_1(\mathbb{RP}^n, [x]) \to \pi_1(S^1, f([x]))$ is the trivial homomorphism.

b) State de definition of a retract $X \subset Y$ of a topological space Y.

c) Let $A \subset \mathbb{RP}^n$ be a subspace that is homeomorphic to S^1. Show that A is not a retract of $\mathbb{RP}^n$.