**Vrije Universiteit Amsterdam** | **27-05-2021 18:45-21:30**
**Final Topology** | **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

**Question 1.** [**5+5+5 Points**] Let $X$ be a topological space and $A \subset X$. Recall that $A$ is a retract of $X$ if there exists a continuous map $r : X \to A$ such that $r\big|_A = \mathrm{id}_A$.

a) Suppose $X = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ and $A = \{(x, y, z) \in X \,|\, z = 0\}$. Show that $A$ is not a retract of $X$.

b) Suppose that $X$ is (path)-connected and $A$ is not. Show that $A$ is not a retract of $X$.

c) Let $x \in A$. Suppose that $\pi_1(X, x)$ is abelian and that $\pi_1(A, x)$ is not abelian. Show that $A$ is not a retract of $X$.

**Question 2.** [**12+8 Points**] Let $X$ be a compact space. Let $C_n \subset X$ with $n \in \mathbb{N}$ be nested, which means that

$$C_1 \supset C_2 \supset \ldots.$$

Assume that each $C_n$ is closed in $X$ and non-empty.

a) Show that $\bigcap_{n\in\mathbb{N}} C_n$ is non-empty.

b) Give an example of a space $X$ (*necessarily not compact!*), and nested closed non-empty subsets $C_n$ such that $\bigcap_{n\in\mathbb{N}} C_n$ is the empty set.

**Question 3.** [**2+10+10 Points** ] Let $X$ be a path-connected space and $x_0, x_1 \in X$. Let $\alpha, \beta : [0, 1] \to X$ be paths from $x_0$ to $x_1$.

a) Explain how the base point changing homomorphism $\widehat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is defined.

b) Show that there exists an element $[\gamma] \in \pi_1(X, x_1)$ such that

$$\widehat{\alpha}([f]) = [\gamma] * \widehat{\beta}([f]) * [\gamma]^{-1}$$

for all $[f] \in \pi_1(X, x_0)$. Give the element $[\gamma]$.

c) Show that $\pi_1(X, x_0)$ is abelian if and only for all $x_0, x_1 \in X$ and all paths $\alpha, \beta$ as above we have that $\widehat{\alpha} = \widehat{\beta}$.

**Continued on the back ⇒ . . .**

**Question 4.** [**3+10+5 Points**] Let $p : E \to B$ be a covering map.

a) Give the definition of a covering map.

Let $f : B' \to B$ be a continuous map. Define the pullback of $E$ along $f$ as

$$f^*E = \{(x, y) \in B' \times E \,|\, y \in p^{-1}(f(x))\}.$$

Define $p' : f^*E \to B'$ by $p'(x, y) = x$. The map $p'$ is a covering map, which you do not need to show. Let $b' \in B'$ and $b = f(b')$.

b) Show that $f^*E$ is not path connected if $f_* : \pi_1(B', b') \to \pi_1(B, b)$ is the trivial homomorphism and $p^{-1}(b)$ contains more than one point.

c) Give an example of a covering map $p : E \to B$, and $f : B' \to B$ such that $E$ is path-connected, but $f^*E$ is not.

**Question 5.** [**15 Points**] Prove Theorem 54.5 from the book: the fundamental group of the circle $S^1$ is isomorphic to the additive group of integers. *You may use properties of the lifting correspondence without proof. State clearly which properties you are using.*