Vrije Universiteit Amsterdam **24-03-2021 8:30-10:15**
Midterm Topology **T.O. Rot**

The use of notes, calculators, etc. is *not* permitted. You may use partial results from previous exercises even if you have not managed to prove them. Explain your answers and state which results you use from the book. Your grade will be $\frac{\text{points}}{10} + 1$.

Question 1. [**3+4+4+3+4+4 Points**] Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X, where the topology $\mathcal{T}'$ is finer than $\mathcal{T}$.

a) State the definition of when $\mathcal{T}'$ is finer than $\mathcal{T}$ and when $(X, \mathcal{T})$ is Hausdorff.

b) Suppose that $(X, \mathcal{T})$ is Hausdorff. Show that $(X, \mathcal{T}')$ is Hausdorff.

c) Give an example of $X, \mathcal{T}, \mathcal{T}'$ as above, where $(X, \mathcal{T}')$ is Hausdorff, but $(X, \mathcal{T})$ is not.

d) State the definition of when $(X, \mathcal{T})$ is connected.

e) Suppose that $(X, \mathcal{T}')$ is connected. Show that $(X, \mathcal{T})$ is connected.

f) Give an example of $X, \mathcal{T}, \mathcal{T}'$ as above, where $(X, \mathcal{T})$ is connected but $(X, \mathcal{T}')$ is not.

Question 2. [**16 Points**] Let $f : X \to Y$ be a continuous and surjective function. Let $A \subset X$ be a dense subspace of X. Recall that this means that $\overline{A} = X$. Show that $f(A)$ is dense in Y.

Question 3. [**8+8+16 Points**] Let (X, d) be a metric space. Define the open ball

$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

and the closed ball

$$C(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}.$$

a) Show that the closure of open balls are contained in the corresponding closed balls. That is, show that

$$\overline{B(x, \epsilon)} \subset C(x, \epsilon).$$

for all $x \in X$ and $\epsilon > 0$.

b) Assume that X has at least two points. The discrete metric on X is the metric

$$\rho(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

Use the metric ρ to show that in general $\overline{B(x, \epsilon)} \neq C(x, \epsilon)$.

c) Show that the following statements are equivalent:

i) For all $x \in X$ and $\epsilon > 0$ we have that $\overline{B(x, \epsilon)} = C(x, \epsilon)$.

ii) For any $x, y \in X$ with $x \neq y$ and $\delta > 0$ there exists z such that $d(z, y) < \delta$ and $d(x, z) < d(x, y)$.

Question 4. [**8+12 Points**] Let X be a Hausdorff topological space and $A \subset X$ be a finite set.

a) Show that A is closed. *We have seen a theorem in the book that states this. You will need to prove this here without quoting that theorem.*

Define the equivalence relation $x \sim y$ if $x = y$ or both $x, y \in A$. We give ${}^{X}/_{\sim}$ the quotient topology.

b) Show that ${}^{X}/_{\sim}$ is Hausdorff.