Time Series Models

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Practice Exam 2021, Including Answers

MC.1 Consider the signal plus noise model for the time series y_t and the signal μ_t, with time index $t = 1, \ldots, n$, as given by

$$y_t = \mu_t + \varepsilon_t, \qquad \mu_{t+1} = \phi\mu_t + \eta_t, \qquad t = 1, \ldots, n,$$

where all variables are scalars, with the autoregressive signal μ_t and the disturbances ε_t and η_t, which are normally and independently distributed with mean zero and variance $\sigma^2_\varepsilon > 0$ and $\sigma^2_\eta > 0$, respectively, both disturbance sequences are mutually and serially uncorrelated, and with autoregressive coefficient $|\phi| < 1$. The initial variable μ_1 is treated as a stochastic variable.

What is the correct expression for the unconditional variance of y_t, that is $\mathbb{V}ar(y_t)$?

a. σ^2_ε

b. $\sigma^2_\eta \,/\, (1 - \phi^2)$

c. $(\sigma^2_\varepsilon + \sigma^2_\eta) \,/\, (1 - \phi^2)$

d. $\sigma^2_\varepsilon + \sigma^2_\eta \,/\, (1 - \phi^2)$

Answer: d.
Since $|\phi| < 1$, we know that μ_t is a stationary AR(1)-process and hence that its unconditional variance is $\sigma^2_\eta/(1-\phi^2)$. Subsequently, $\mathbb{V}ar(y_t) = \mathbb{V}ar(\mu_t) + \mathbb{V}ar(\varepsilon_t) = \sigma^2_\eta/(1 - \phi^2) + \sigma^2_\varepsilon$.

MC.2 Suppose x and y are jointly Normally distributed vectors with means $\mu_x = \mathbb{E}(x)$ and $\mu_y = \mathbb{E}(y)$, respectively, with variance matrices Σ_{xx} and Σ_{yy}, respectively, and covariance matrix Σ_{xy}. Further define $\widehat{x} = \mathbb{E}(x|y)$ and $e = x - \widehat{x}$.

Can you provide an expression for $\mathbb{E}(xe'|y)$ *?*

a. 0

b. $\mathbb{C}ov(x, y)$

c. $\mathbb{V}ar(x|y)$

d. $\mathbb{V}ar(x)$

Answer: c.
The derivation is very much similar to Slide 23 of Tutorial 2. In this case, step 1 will give $\mathbb{E}[\hat{x}e'|y] = 0$ and step 2 uses $\mathbb{E}[xe'|y] = \mathbb{E}[xe'|y] - \mathbb{E}[\hat{x}e'|y]$ for $e = x - \widehat{x}$. The only difference with the mentioned slide is that we already condition on the vector y, hence the LIE and (B) are no longer needed.

MC.3 Consider the signal plus noise model for the time series y_t and the signal μ_t, with time index $t = 1, \ldots, n$, as given by

$$y_t = \mu_t + \varepsilon_t, \qquad \mu_{t+1} = \phi\mu_t + \eta_t, \qquad t = 1, \ldots, n,$$

where all variables are scalars, with the autoregressive signal μ_t and the disturbances ε_t and η_t, which are normally and independently distributed with mean zero and variance $\sigma^2_\varepsilon > 0$ and $\sigma^2_\eta > 0$, respectively, both disturbance sequences are mutually and serially uncorrelated, and with autoregressive coefficient $|\phi| < 1$. The initial variable μ_1 is treated as a stochastic variable.

We need to apply the Kalman filter for the model in state space form with state vector $\alpha_t \equiv \mu_t$. For the parameters of the model, we take the values $\phi = 0.8$, $\sigma^2_\varepsilon = 1$ and $\sigma^2_\eta = 0.52$. For a given value of $a_1 = 1$ and $p_1 = 3$, and for the realised observations $y_1 = 2$ and $y_2 = 1$.

Please provide the numerical value for $a_3 = \mathbb{E}(\alpha_3 | y_1, y_2)$.

a. $a_3 = -0.12$

b. $a_3 = 0.96$

c. $a_3 = 1.33$

d. $a_3 = -2.44$

Answer: b.
We use Slide 24 of Lecture 2 for the AR(1)+noise model. This gives $f_1 = 4, k_1 = 0.6$ and $v_1 = 1$ such that $a_2 = 1.4$ and $p_2 = 1$. Then, $f_2 = 2, k_2 = 0.4$ and $v_2 = -0.4$ such that $a_3 = 0.96$.

MC.4 Consider a bivariate dynamic factor model for the time series x_t and y_t as given by

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 \\ \beta \end{pmatrix} \mu_t + \varepsilon_t, \qquad \mu_{t+1} = \mu_t + \eta_t,$$

where μ_t is the common random effect modeled as a random walk process, β is the loading (or weight) to the common effect for y_t, the disturbance vector $\varepsilon_t \sim NID(0, \Sigma_\varepsilon)$ and disturbance $\eta_t \sim NID(0, \sigma_\eta^2)$ are normally distributed, serially and mutually independent, for $t = 1, \ldots, n$. The 2×2 variance matrix Σ_ε and the variance σ_η^2 are both unknown.

Your colleague is interested in the time series variable z_t that is of key interest. She has derived the following structural equation for z_t, that is

$$z_t = a + 5 \cdot x_t + 2 \cdot y_t + \kappa_t \qquad \kappa_t \sim NID(0, \sigma_\kappa^2),$$

where a is an unknown scalar constant and σ_κ^2 is the unknown variance of the disturbances κ_t, and where x_t and y_t are generated by the dynamic factor model given above. Your colleague argues that z_t can be assumed stationary.

What condition for β is needed to assume that z_t is stationary ?

a. $\beta = 2.5$

b. $\beta = -0.4$

c. $\beta = -2.5$

d. $\beta = 0.4$

Answer: c.
To have that z_t is a stationary process, we need that the random walk process μ_t is no longer part of it. By rewriting the expression of z_t, we see that $z_t = a + (5 + 2\beta)\mu_t + 5\varepsilon_{1t} + 2\varepsilon_{2t} + \kappa_t$. Hence, z_t can only be stationary if $5 + 2\beta = 0$.

MC.5 Consider the non-Gaussian state space model

$$y_t \sim p(y_t|\theta_t), \qquad \theta_t = Z_t\alpha_t, \qquad \alpha_{t+1} = d_t + T_t\alpha_t + \eta_t,$$

for $t = 1, \dots, n$, with the usual notation. In our analysis of the founder's day in March of the First Peewit Egg, we analyse the resulting yearly time series y_t using the exponential density as given by

$$p(y_t|\theta_t) = \exp(-\theta_t)\exp\left[-y_t \exp(-\theta_t)\right], \qquad y_t > 0.$$

To compute the conditional mode of θ_t, for $t = 1, \dots, n$, we adopt a linear model $x_t = \theta_t + u_t$ with error $u_t \sim NID(0, A_t)$.

What is the expression for A_t ?

a. $A_t = 1$

b. $A_t = 1/y_t$

c. $A_t = \exp(\theta_t)/y_t$

d. $A_t = \exp(\theta_t)$

Answer: c.
We follow the usual notation of Slides 51 and 54 of Week 4. First consider $\log p(y_t|\theta_t) = -\theta_t - y_t \exp(-\theta_t)$, such that $\dot{p}_t = -1 + y_t \exp(-\theta_t)$ and $\ddot{p}_t = -y_t \exp(-\theta_t)$. Then we know that $A_t = -\ddot{p}_t^{-1} = \exp(\theta_t)/y_t$.

Open Question

Consider the signal plus noise model for time series y_t and signal μ_t as given by

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, & & & \varepsilon_t &\sim NID(0, \sigma_\varepsilon^2), \\ \mu_{t+1} &= \mu_t + \beta_t, & \beta_{t+1} &= \phi\, \beta_t + \zeta_t, & \zeta_t &\sim NID(0, \sigma_\zeta^2), \end{aligned} \qquad t \in \mathbb{Z},$$

where all variables are scalars: signal μ_t is a "smooth" trend process with a stationary autoregressive process for the growth β_t with autoregressive coefficient $|\,\phi\,| < 1$. The disturbances ε and ζ_t are normally and independently distributed, both disturbances are also mutually independent. The parameters in the model are collected in vector $\psi = (\phi\,,\, \sigma_\varepsilon^2\,,\, \sigma_\zeta^2)'$. We have two observations y_1 and y_2, that is $T = 2$.

Give an expression for the exact diffuse loglikelihood (with $\kappa \to \infty$) as a function in terms of observations and parameter vector ψ, that is, function of y_1, y_2, ϕ, σ_ε^2 and σ_ζ^2.

HINT: start with placing the model in state space form, go through the Kalman filter equations for $t = 1, 2$, and construct the expression for the *diffuse* loglikelihood function.

Answer:

State space

$$\alpha_t = (\mu_t, \beta_t)', \quad Z_t = (1,0), \quad H_t = \sigma_\varepsilon^2, \quad T_t = \begin{bmatrix} 1 & 1 \\ 0 & \phi \end{bmatrix}, \quad R_t = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q_t = \sigma_\zeta^2,$$

with initial conditions

$$a_1 = 0, \qquad P_1 = \begin{bmatrix} \kappa & 0 \\ 0 & \sigma_\zeta^2 \,/\, (1-\phi^2) \end{bmatrix}, \qquad \kappa \to \infty.$$

Kalman filter $t = 1$: define $p_* = \sigma_\zeta^2 \,/\, (1-\phi^2)$ such that

$$\begin{aligned} v_1 &= y_1 \\ f_1 &= \kappa + \sigma_\varepsilon^2, \qquad \kappa \to \infty \\ K_1 &= \begin{bmatrix} 1 & 1 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} \kappa & 0 \\ 0 & p_* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} /(\kappa + \sigma_\varepsilon^2) \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

Kalman update:

$$\begin{aligned} a_2 &= T_1 a_1 + K_1 v_1 = 0 + (1,0)' y_1 = (y_1, 0)' \\ P_2 &= \begin{bmatrix} 1 & 1 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} \kappa & 0 \\ 0 & p_* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & \phi \end{bmatrix} - K_1 f_1 K_1' + R_1 Q_1 R_1' \\ &= \begin{bmatrix} \kappa + p_* & \phi p_* \\ \phi p_* & \phi^2 p_* \end{bmatrix} - \begin{bmatrix} \kappa^2/(\kappa + \sigma_\varepsilon^2) & 0 \\ 0 & 0 \end{bmatrix} + R_1 Q_1 R_1' \\ &\to p_* \begin{bmatrix} 1 & \phi \\ \phi & \phi^2 \end{bmatrix} + \begin{bmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \sigma_\zeta^2 \end{bmatrix}. \end{aligned}$$

Computing v_2 and f_2:

$$\begin{aligned} v_2 &= y_2 - y_1 \\ f_2 &= p_* + 2\sigma_\varepsilon^2 \;=\; [\sigma_\zeta^2 \,/\, (1-\phi^2)] + 2\sigma_\varepsilon^2 \end{aligned}.$$

Diffuse loglikelihood function definition:

$$\ell_D(y_1, y_2; \psi) = \ell(y_1, y_2; \psi) + \frac{1}{2} \log \kappa,$$

as $\kappa \to \infty$. We have

$$\ell_D(y_1, y_2; \psi) = -\log 2\pi - \frac{1}{2} \log f_1 - \frac{1}{2} v_1^2 / f_1 - \frac{1}{2} \log f_2 - \frac{1}{2} v_2^2 / f_2 + \frac{1}{2} \log \kappa,$$

where $v_1 = y_1$, $f_1 = \kappa + \sigma_\varepsilon^2$, $v_2 = y_2 - y_1$ and $f_2 = [\sigma_\zeta^2 \,/\, (1-\phi^2)] + 2\sigma_\varepsilon^2$.

Final expression: first notice

$$\log f_1 = \log \kappa (f_1/\kappa) = \log \kappa + \log(f_1/\kappa).$$

Then we obtain

$$\ell_D(y_1, y_2; \psi) = -\log 2\pi - \frac{1}{2} \log(f_1/\kappa) - \frac{1}{2} y_1^2/(\kappa + \sigma_\varepsilon^2) - \frac{1}{2} \log f_2 - \frac{1}{2} (y_2 - y_1)^2 / f_2,$$

as $\kappa \to \infty$,

$$\ell_D(y_1, y_2; \psi) = -\log 2\pi - \frac{1}{2} \log \left([\sigma_\zeta^2 \,/\, (1-\phi^2)] + 2\sigma_\varepsilon^2 \right) - \frac{1}{2} (y_2 - y_1)^2 / \left([\sigma_\zeta^2 \,/\, (1-\phi^2)] + 2\sigma_\varepsilon^2 \right).$$