vrije Universiteit amsterdam

Time Series Econometrics 4.4

MSc Econometrics
Faculty of Economics and Business Administration
Thursday, March 24, 2016

Exam:	Time Series Econometrics 4.4
Code:	E_EORM_TSE
Examinator:	prof.dr. S.J. Koopman
Co-reader:	Dr. C.S. Bos
Date:	March 24, 2016
Room:	WN–KC137
Time:	15:15
Duration:	2:45 hours
Calculator:	Allowed
Graphical calculator:	Allowed
Number of questions:	4
Type of questions:	Open
Answer in:	English
Credit score:	100 credits counts for a 10
Grades:	made public within 10 working days
Inspection:	Tuesday, April 6, 2016 at 11.00, 11th floor
Number of pages:	5, including front page

- Read the entire exam carefully before you start answering the questions.
- Be clear and concise in your statements.
- The questions should be handed back at the end of the exam. Do not take it home.

Good luck!

Question 1 [30 points] Local level model

Consider the local level model for the time series observations $y_1, \ldots, y_n$ as given by

$$y_t = \mu_t + \varepsilon_t, \qquad \mu_{t+1} = \mu_t + \eta_t, \qquad t = 1, \ldots, n,$$

where all variables are scalars, with the level μ_t and where the disturbances ε_t and η_t are normally distributed with mean zero and variance $\sigma_\varepsilon^2 > 0$ and $\sigma_\eta^2 > 0$, respectively, and are mutually and serially uncorrelated. The initial level μ_1 is treated as an unknown variable. The signal-to-noise ratio is defined as $q = \sigma_\eta^2 \,/\, \sigma_\varepsilon^2$.

(a) Discuss the initial (unconditional) distribution for μ_1, give its mean a_1 and variance p_1, and motivate your answer for the mean and variance.

(b) When you decide to estimate μ_1 by maximum likelihood, together with σ_ε^2 and σ_η^2, and hence the parameter vector is $\psi = (\mu_1 \,,\, \sigma_\varepsilon^2 \,,\, \sigma_\eta^2)'$, how would you initialize the Kalman filter ? In other words, what are your values for a_1 and p_1 in this case ? Motivate your answer.

(c) The Kalman filter for the local level is given by

$$\begin{aligned} v_t &= y_t - a_t, & f_t &= p_t + \sigma_\varepsilon^2, \\ & & k_t &= p_t / f_t, \\ a_{t+1} &= a_t + k_t v_t, & p_{t+1} &= p_t(1 - k_t) + \sigma_\eta^2, \end{aligned}$$

for $t = 1, \ldots, n$ and initial values a_1 and p_1. Give the definitions for v_t and f_t, derive their expressions and motivate your steps in the derivations.

(d) Show that we can express the Kalman update equations as

$$a_{t+1} = (1 - k_t)a_t + k_t y_t, \qquad p_{t+1} = \sigma_\varepsilon^2(k_t + q), \qquad t = 1, \ldots, n.$$

(e) Develop an algorithm for evaluating the conditional mean and variance

$$a_t^* = \mathbb{E}(\mu_t | y_1, \ldots, y_t, y_{t+1}; \psi), \qquad p_t^* = \mathbb{V}\mathrm{ar}(\mu_t | y_1, \ldots, y_t, y_{t+1}; \psi), \qquad t = 1, \ldots, n.$$

[Note: it is not a typo, we condition on $y_1, \ldots, y_{t+1}$, NOT on $y_1, \ldots, y_{t-1}$]. Give a clear description of your algorithm.

Question 2 [25 points] Multivariate local level model

The multivariate local level model is given by

$$y_t = \mu_t + \varepsilon_t, \qquad \mu_{t+1} = \mu_t + \eta_t, \qquad t = 1, \ldots, n,$$

where all variables are $p \times 1$ vectors, with the level μ_t and where the disturbances ε_t and η_t are normally distributed vectors with mean zero and $p \times p$ variance matrices Σ_ε and Σ_η, respectively, and which are mutually and serially uncorrelated.

(a) Show that the cross-sectional average $\bar{y}_t = p^{-1} \sum_{i=1}^{p} y_{it}$, where y_{it} is the i-th element of y_t, for $t = 1, \ldots, n$, can be represented by a local level model. Provide the details with definitions of the variables in the model AND an expression for its signal-to-noise ratio q.

(b) The rank of Σ_η is restricted to be equal to unity, that is $\text{rank}(\Sigma_\eta) = 1$. Give an alternative expression for the multivariate local level model of y_t in terms of the scalar random walk level $\mu_t^\dagger$, that is $\mu_{t+1}^\dagger = \mu_t^\dagger + \eta_t^\dagger$. Provide all details of your model specification, including the initial conditions. Also show that your specification is equivalent to the multivariate specification above.

(c) For the multivariate local level model with $\text{rank}(\Sigma_\eta) = 1$, it is your suggestion to estimate $\mu_t^\dagger$ via the Kalman filter that is applied to a scalar time series y_t^* and based on a scalar local level model. How would you transform (linearly) the $p \times 1$ observation vector y_t to obtain y_t^* and from which you can apply the Kalman filter to estimate $\mu_t^\dagger$? Provide all details of your solution.

(d) In the previous sub-question (c), can you also consider $y_t^* = \bar{y}_t$? Motivate your answer and provide the details.

Question 3 [25 points] Importance Sampling

Consider the time-varying regression model for the univariate time series y_t,

$$y_t = X_t\beta_t + \varepsilon_t, \qquad \beta_{t+1} = \Phi\beta_t + \eta_t, \qquad t = 1, \ldots, n,$$

where disturbances ε_t and η_t are mutually and serially uncorrelated, X_t is a $1 \times k$ vector of covariates, β_t is a time-varying $k \times 1$ vector of coefficients, for $t = 1, \ldots, n$. The time-varying regression parameter is specified as a vector autoregressive process, a VAR(1) process with coefficient matrix Φ. The disturbance ε_t is from the Student's t distribution and η_t is from the normal distribution. The Student's t distribution has logdensity

$$\log p(\varepsilon_t) = \log a(\nu) + \frac{1}{2}\log\lambda - \frac{\nu+1}{2}\log(1+\lambda\varepsilon_t^2),$$

where $\nu > 2$ is the degrees of freedom and

$$a(\nu) = \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu/2)}, \qquad \lambda^{-1} = (\nu-2)\sigma_\varepsilon^2, \qquad \sigma_\varepsilon^2 = \mathrm{Var}(\varepsilon_t),$$

for $t = 1, \ldots, n$.

(a) Provide the details for the iterative procedure to compute the mode of $p(\beta_t|Y_n;\psi)$ with $Y_n = \{y_1, \ldots, y_n\}$ and where vector of unknown coefficients ψ is assumed known, for $t = 1, \ldots, n$. It is assumed that you know the general algorithm for computing the mode; you only need to provide the specific details for the Student's t density.

(b) Derive an expression for the Monte Carlo estimate of the likelihood function using importance sampling.

(c) The Monte Carlo estimate of the likelihood function is consistent. A central limit theorem does apply when the variance exists for the importance weights. How would you verify this condition ?

(d) The economist is interested in the signal f_t which is a specific function of the regression coefficients, that is $f_t = f(\beta_t)$ for some continuous function $f()$, at time $t = n$. Describe how you would construct a Monte Carlo estimate of f_n using importance sampling.

Question 4 [20 points] Nonlinear filtering

Consider a nonlinear unobserved components time series model with trend μ_t and autoregressive component ψ_t. The observation equation is given by

$$y_t = \mu_t + \psi_t, \qquad t = 1, \ldots, n,$$

where μ_t is specified as the random walk process given by

$$\mu_{t+1} = \mu_t + \eta_t,$$

and where ψ_t is specified as the autoregressive process given by

$$\psi_{t+1} = \Phi(\mu_t)\psi_t + \varepsilon_t,$$

with $\Phi(\cdot)$ being the cumulative probability density function for some standardized normal variable. The disturbances ε_t and η_t are normally distributed with mean zero and variance $\sigma^2_\varepsilon > 0$ and $\sigma^2_\eta > 0$, respectively, and are mutually and serially uncorrelated.

(a) Express the model in the form of a general nonlinear state space model

$$y_t = Z(\alpha_t), \qquad \alpha_{t+1} = T(\alpha_t) + \xi_t,$$

where α_t is the state vector and ξ_t is the corresponding disturbance vector. Give expressions for α_t and ξ_t, and for the functions $Z(\alpha_t)$ and $T(\alpha_t)$, in terms of the model specification given above.

(b) Construct the Extended Kalman filter for this nonlinear model. You can treat the standardized probability density function $\phi(z) = \partial\Phi(z) \,/\, \partial z$ as given, without providing an expression for it.

(c) An alternative to the Extended Kalman filter is the particle filter. What is the merit of adopting the particle filter in this case ? In other words, why would you advocate to use the particle filter, rather than the extended Kalman filter ?

Note : the standard Kalman filter equations for the linear Gaussian state space model $y_t = Z_t \cdot \alpha_t + \epsilon_t$ and $\alpha_{t+1} = T_t \cdot \alpha_t + R_t\zeta_t$ are given by

$$\begin{aligned} v_t &= y_t - Z_t a_t, & F_t &= Z_t P_t Z_t' + H_t, \\ & & K_t &= T_t P_t Z_t' F_t^{-1}, \\ a_{t+1} &= T_t a_t + K_t v_t, & P_{t+1} &= T_t P_t T_t' - K_t F_t K_t' + R_t Q_t R_t', \end{aligned}$$

for $t = 1, \ldots, n$ and initial values a_1 and p_1, where H_t and Q_t are the variance matrices for ϵ_t and ζ_t, respectively.