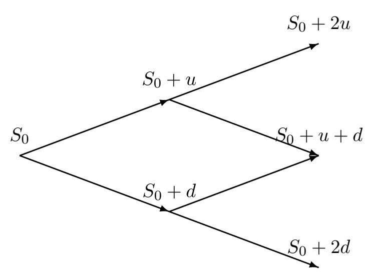
Stochastic Processes: The Fundamentals

Exam

October 27, 2021

Grading: you can earn 90 points with this exam. Your exam grade is then: $\frac{\#\text{points}}{10} + 1$. The exam counts for 60% of your final grade.

Question 1: Binomial trees (23 points)
We want to price options on the S&P-500 index. The remaining time to maturity of the options is one year. Assume that the S&P-500 index pays no dividends. We model the evolution of the S&P-500 index by means of a **2-step binomial tree** where the two steps have equal length of **6 months**. The probability of an upward movement is p. The current value of the S&P-500 index, which we denote as S_0, is 4,500. The risk free interest rate r is -0.20% per annum with semi-annual compounding (no, it is not a typo: we assume a negative risk free interest rate). The picture below provides a graphical view of the process that we assume for the S&P-500 index.



Assume that $u = 600$.

(a) Determine p and d and provide the resulting tree, where the following conditions should be satisfied (**5 pts**):

- $d = -u$
- the expected 1-year **simple net return** on the S&P-500 index is 3%.

Note: if S_0 = 4,500 and S_1 = 4,950, then the **simple net return** equals 4,950/4,500 - 1 = 10%.

Solution:
We know that $d = -u$ and hence $d = -600$. This leads to the following net simple returns:

- 26.67% in node 1 (with probability p^2)

- 0% in node 2 (with probability $2p(1-p)$)
- -26.67% in node 3 (with probability $(1-p)^2$)

In order to find the value for p we solve the following equation:

$$p^2 \cdot 26.67\% + (1-p)^2 \cdot (-26.67\%) = 3\%.$$

This is equivalent to solving for p in the following equation:

$$(1-2p) \cdot (-26.67\%) = 3\%.$$

Solving this equation gives $p = 55.625\%$.

(b) Use the binomial tree and the risk-neutral valuation method to calculate the no-arbitrage price of a **European put-option** on the S&P-500 index with strike price 4,500 and a remaining time to maturity of one year. If you were not able to solve (a) then choose $d = -u$. (**10 pts**)

Solution:
The tricky thing in this exercise is that interest rates are non-zero. This makes the upward risk-neutral probabilities state-dependent. A simple approach is to apply the formulas of the slides to each relevant one-period binomial tree.

The put-option only provides a payoff in the bad scenario. This payoff equals USD 1,200. The risk-neutral probability of this event happening conditional on the S&P-500 index equal to 3,900 after 6 months is:

$$q_1^d | S_1^d = \frac{4,500/3,900 - (1-0.10\%)}{4,500/3,900 - 3,300/3,900} = 50.325\%.$$

Doing the same for the first period results in $q_0^d = 50.375\%$.

Now we can calculate the no-arbitrage price of the European put-option as follows:

$$\frac{1}{(1-0.1\%)^2} \cdot 50.375\% \cdot 50.325\% \cdot 1,200 = 304.824.$$

(c) Provide an expression for $\mathbb{E}^{\mathbb{Q}}\left(S_{1\text{ year}} | S_{6\text{ months}}\right)$, i.e. what is the risk-neutral expectation of the stock price in 1 year conditional on knowledge about the stock price in 6 months (**3 pts**).

Solution:
Under the risk-neutral probability all assets earn the risk-free rate in expectation:

$$\mathbb{E}^{\mathbb{Q}}\left(S_{1\text{ year}} | S_{6\text{ months}}\right) = 5,100 \cdot (1-0.10\%) \cdot \mathbb{I}_{\{S_{6m}=5,100\}} + 3,900 \cdot (1-0.10\%) \cdot \mathbb{I}_{\{S_{6m}=3,900\}}.$$

(d) Use the binomial tree and the risk-neutral valuation method to calculate the no-arbitrage price of an **American call-option** on the S&P-500 index with strike price 4,400 and a remaining time to maturity of one year (**5 pts**).

Solution:
Value in S_1^u is given by:

$$\max\left(5,100-4,400, \frac{1}{1-0.1\%}(0.49575\cdot 1,300+(1-0.49575)\cdot 100)\right)=700.$$

Value in S_1^d is given by:

$$\frac{1}{1-0.1\%}(0.49675\cdot 100+(1-0.49675)\cdot 0)=49.725.$$

The value of the American call option is:

$$\frac{1}{1-0.1\%}(0.49625\cdot 700+(1-0.49625)\cdot 49.725)=372.797.$$

Question 2: Equity derivatives (12 pts)
A US-based institutional investor becomes aware of a new contract in the market, the so-called **return contract on the S&P-500 index**. If the investor takes a long position in this contract, the payoff at the maturity date is the excess realized return on the S&P-500 index between inception date and maturity date of the contract. Assume that the per annum US risk free interest rate is 1% (with annual compounding). Let's illustrate the return contract by means of an example: if an investor enters into a long position of a 1-year contract with a notional amount of USD 100,000,000 and the realized 1-year simple net return on the S&P-500 index is 5%, then the investor receives after 1 year:

$$\text{USD } 100,000,000\times(5\%-1\%)=\text{USD } 4,000,000.$$

Hence, the 1-year return contract pays the 1-year return of the S&P-500 index in excess of the 1-year risk free rate on the traded notional amount.

The US-based institutional investor is interested in the return contract and asks a price quote in the market. It appears that the price for taking a long position in the 1-year return contract is 0.10% of the notional amount. If the investor wants to go short in the 1-year return contract, she receives 0.10% of the notional. After carefully analyzing the contract and its price, the investor concludes that the price of the return contract offers an arbitrage opportunity. Describe in detail the strategy the institutional investor should employ in order to earn a riskless profit. You can assume frictionless markets, a per annum risk free interest rate of 1% (with annual compounding) and a notional amount of USD 100,000,000.

Solution:
We first calculate the no-arbitrage price of this contract by means of the risk-neutral valuation method:

$$V(0)=\frac{1}{1+1\%}\mathbb{E}_0^{\mathbb{Q}}\left(N\cdot\left(\frac{S(1)}{S(0)}-1-1\%\right)\right)=\frac{1}{1+1\%}\cdot N\cdot(1+1\%-1-1\%)=0.$$

Arbitrage strategy:

- Short the return contract and receive USD 100,000
- Invest this amount for a period of 1 year in the risk-free asset
- Borrow USD 100,000,000 for 1 year against 1% and invest this amount in the S&P-500 index
- After 1 year sell the position in the S&P-500 index and deliver the excess return to your counterparty.
- Pay back your creditor USD 100,000,000 plus interest
- Whatever happens with the S&P-500 index, you are left with USD $100,000 \times (1+1\%)$.

Question 3: Itô's Lemma (20 pts)
Let $W = (W_t)_{t\geq 0}$ be a Brownian Motion process. Define the process Y as:

$$Y_t := t^2 W_t^3, t \geq 0.$$

(a) Derive the stochastic differential equation for Y (**10 pts**). Make sure that in the final expression the drift and diffusion terms are written as function of Y_t, i.e.

$$dY_t = f(Y_t)dt + g(Y_t)dW_t.$$

Solution:
Apply Itô's Lemma:

$$\begin{aligned} dY_t &= 2tW_t^3 dt + 3t^2 W_t^2 dW_t + 3t^2 W_t dt \\ &= \left(2tW_t^3 + 3t^2 W_t\right) dt + 3t^2 W_t^2 dW_t \\ &= (2Y_t/t + 3(t^4 Y_t)^{1/3})dt + 3(tY_t)^{2/3} dW_t. \end{aligned}$$

Let $W = (W_t)_{t\geq 0}$ again be a Brownian Motion process. Define the process X as:

$$X_t := e^{-\alpha t}\left(X_0 + \sigma \int_0^t e^{\alpha s} dW_s\right).$$

(b) Derive the stochastic differential equation for X. (**10 pts**)

Solution:

$$\begin{aligned} dX_t &= d\left(e^{-\alpha t}\left(X_0 + \sigma \int_0^t e^{\alpha s} dW_s\right)\right) \\ &= e^{-\alpha t}\sigma e^{\alpha t} dW_t - \alpha e^{-\alpha t}\left(X_0 + \sigma \int_0^t e^{\alpha s} dW_s\right) dt \\ &= \sigma dW_t - \alpha X_t dt. \end{aligned}$$

Question 4: Change of Measure (15 pts)
Let S be a martingale satisfying the stochastic differential equation $dS_t = \sigma S_t dW_t$, starting from $S_0 = 1$, where σ is a strictly positive constant and W a Brownian Motion under the probability measure $\mathbb{P}$.

(a) Show that the stochastic differential equation for the stochastic process $X := S^{-1}$ equals (**5 pts**):

$$dX_t = -\sigma X_t dW_t + \sigma^2 X_t dt.$$

Solution:

$$\begin{aligned} dX_t &= -\frac{1}{S^2} dX_t + \frac{1}{2} \cdot 2 \cdot \frac{1}{S^3} (dX_t)^2 \\ &= -\frac{1}{S^2} \sigma S dt + \frac{1}{S^3} \sigma^2 S^2 dt \\ &= \sigma^2 X_t dt - \sigma X_t dW_t. \end{aligned}$$

Now we want to define a new probability measure $\mathbb{Q}$ and we want to do this in such a way that X becomes a martingale under $\mathbb{Q}$.

(b) Provide the SDE for stochastic process X under the probability measure $\mathbb{Q}$ by applying Girsanov's theorem. Also provide the Radon-Nikodym derivative process Z, in integral form, that connects the probability measures $\mathbb{P}$ and $\mathbb{Q}$. (**10 pts**)

Solution:

$$dX_t = -\sigma X_t (dW_t - \sigma dt).$$

Using Girsanov's theorem we can define

$$\tilde{W}_t = W_t - \int_0^t \sigma du.$$

This process is Brownian Motion under a probability measure $\mathbb{Q}$. The process for X under $\mathbb{Q}$ is given by:

$$dX_t = -\sigma X_t d\tilde{W}_t.$$

The Radon-Nikodym derivative process Z that connects the two probability measures is given by:

$$Z_t = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}.$$

Question 5: Payoff pricing (20 pts)
Assume the following process for the stock price:

$$dS_t = \mu dt + \sigma dW_t,$$

where $\mu > 0$, $\sigma > 0$ and W is a Brownian Motion under the real-world probability measure $\mathbb{P}$.

The second asset in this economy is a risk free asset, B. The process for B is given by the following differential equation:

$$dB_t = rB_t dt,$$

where $r < \mu$ stands for the constant continuously compounded risk free interest rate.

(a) Derive the process for S, in differential form, under the risk-neutral probability measure $\mathbb{Q}$. (**6 pts**)

Solution: First, derive the process for the discounted stock price:

$$\begin{aligned} d\left(\frac{S_t}{B_t}\right) &= \frac{1}{B_t} dS_t - \frac{S_t}{B_t^2} dB_t \\ &= \frac{1}{B_t}(\mu dt + \sigma dW_t) - r\frac{S_t}{B_t} dt \\ &= \frac{\sigma}{B_t}\left(dW_t + \left(\frac{\mu - rS_t}{\sigma}\right) dt\right). \end{aligned}$$

Applying Girsanov:

$$d\left(\frac{S_t}{B_t}\right) = \frac{\sigma}{B_t} d\tilde{W}_t.$$

Now enter the definition for $\tilde{W}_t$ in the equation for S:

$$\begin{aligned} dS_t &= \mu dt + \sigma dW_t \\ &= \mu dt + \sigma\left(d\tilde{W}_t - \left(\frac{\mu - rS_t}{\sigma}\right) dt\right) \\ &= rS_t dt + \sigma d\tilde{W}_t. \end{aligned}$$

Under the risk-neutral probability measure $\mathbb{Q}$, the random variable $S_t|S_0$ has a **normal distribution** of which the mean and the variance can be derived from the solution of the stochastic differential equation for S under $\mathbb{Q}$:

$$S_t = e^{rt} S_0 + \sigma e^{rt} \int_0^t e^{-rs} d\tilde{W}_s.$$

Assume $0 < \alpha < \beta$. At time t we want to find the no-arbitrage price of a contract that pays off K if S at the maturity date T is below α and pays off $2K$ if S at the maturity date T is above β. The derivative doesn't pay anything if S_T is in $[\alpha, \beta]$.

(b) Derive the time t no-arbitrage price of this payoff structure in the economy that is described at the start of this exercise. (**14 pts**)

Solution:

The value $V(t)$ can be found by:

$$\begin{aligned} V(t) &= e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}(V(T)) \\ &= e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}(K\mathbb{I}_{\{S_T < \alpha\}} + 2K\mathbb{I}_{\{S_T > \beta\}}) \\ &= e^{-r(T-t)} K\left(\mathbb{Q}_t(S_T < \alpha) + 2(1 - \mathbb{Q}_t(S_T \leq \beta))\right). \end{aligned}$$

We need the distribution of S_T under the risk-neutral probability measure. We can find the variance by applying Ito isometry:

$$S_T|S_t \sim N\left(e^{r(T-t)}S_t, \frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)\right).$$

We can use this information to derive the no-arbitrage price:

$$\begin{aligned}
V(t) &= e^{-r(T-t)}K\left(\mathbb{Q}_t(S_T < \alpha) + 2(1-\mathbb{Q}_t(S_T \leq \beta)\right) \\
&= e^{-r(T-t)}K\left(\mathbb{Q}_t\left(\frac{S_T - e^{r(T-t)}S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)}} < \frac{\alpha - e^{r(T-t)}S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)}}\right)\right. \\
&\left. + 2(1-\mathbb{Q}_t\left(\frac{S_T - e^{r(T-t)}S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)}} < \frac{\beta - e^{r(T-t)}S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)}}\right)\right)\right) \\
&= e^{-r(T-t)}K\left(\Phi\left(\frac{\alpha - e^{r(T-t)}S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)}}\right) + 2\left(1-\Phi\left(\frac{\beta - e^{r(T-t)}S_t}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)}-1\right)}}\right)\right)\right).
\end{aligned}$$